UNIT 3 MATRICES - II

Structure

3.0 INTRODUCTION

In Unit 2, we have introduced Matrices. In this Unit, we shall study elementary operation on Matrices. There are basically three elementary operations. Scaling, Interchange and Replacement. These operations are called elementary row operations or elementary column operations according as they are performed on rows and columns of the matrix respectively. Elementary operations play important role in reducing Matrices to simpler forms, namely, triangular form or normal form. These forms are very helpful in finding rank of a matrix, inverse of a matrix or in solution of system of linear equations. Rank of a matrix is a very important concept and will be introduced in this unit. We shall see that rank of a matrix remains unaltered under elementary row operations. This provides us with a useful tool for determining the rank of a givne matrix. We have already defined inverse of a square matrix in Unit 2 and discussed a method of finding inverse using adjoint of a matrix. In this unit, we shall discuss a method of finding inverse of a square matrix using elementary row operations only.

3.1 OBJECTIVES

After studying this Unit, you should be able to :

- define elementary row operations;
- reduce a matrix to triangular form using elementary row operations;
- reduce a matrix to normal form using elementary operations;
- define a rank of a matrix;
- find rank of a matrix using elementary operations;
- find inverse of a square matrix usng elementary row operations.

3.2 ELEMENTARY ROW OPERATIONS

Consider the matrices of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$ and $D= \begin{bmatrix} 9 & 12 & 15 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Matrices B, C and D are related to the matrix A as follows :

- Matrix B can be obtained from A by multiplying the first row of A by 2;
- Matrix C can be obtained from A by interchanging the first and second rows;
- Matrix D can be obtained from A by adding twice the second row the first row.

Such operations on the rows of a matrix are called elementary operations.

Definitions : An elementary row operations is an operation of any one of the following three types :

1. **Scaling :** Multiplication of a row by a non zero constant.
2. **Interchange :** Interchange of two rows.
3. **Replacement :** Adding one row to a multiple of another row.

We denote scaling by $R_i \rightarrow kR_i$, interchange by $R_i \leftrightarrow R_j$ and replacement by $R_i \rightarrow R_i + kR_j$.

Thus, the matrices B, C and D are obtained from matrix A by applying elementary row operations $R_1 \rightarrow 2R_1$ $R_1 \leftrightarrow R_2$ and $R_1 \rightarrow R_1 + 2R_2$ respectively.

Definiton : Two matrices A and B are said to be row equivalent, denoted by $A \sim B$, if one can be obtained from the other by a finite sequence of elementary row operations.

Clearly, matrices B, C and D discussed above are row equivalent to the matrices A and also to each other by the following remark.

Remark : If A, B and C are three matrices, then the following is obvious.

1. $A \sim A$
2. If $A \sim B$, then $B \sim A$
3. If $A \sim B$, $B \sim C$, then $A \sim C$.

Example 1 : Show that matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is row equivalent to the matrix.

B = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$

Solution : We have A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Applying $R_2 \to R_2 - 4\,R_1$, we have

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$$

Applying $R_3 \to R_3 - 7\,R_1$ to the matrix on R. H. S. we get.

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$

Now Applying $R_3 \to R_3 - 2\,R_2$, we have

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} = B$$

The matrix B in above example is a triangular matrix.

Definition : A matrix A = $[a_{ij}]$ is called a triangular matrix if $a_{ij} = 0$ whenver $i > j$.

In the above example, we reduced matrix A to the triangular matrix B by elementary row operations. This can be done for any given matrix by the following theorem that we state without proof.

Theorem : Every matrix can be reduced to a triangular matrix by elementary row operations.

Example 2 : Reduce the matrix

A = $\begin{bmatrix} 5 & 3 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & 2 & 0 \end{bmatrix}$

to triangular form.

Solution : A = $\begin{bmatrix} 5 & 3 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & 2 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 1 & 2 \end{bmatrix} \qquad \text{(by applying } R_1 \leftrightarrow R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & -9 & 0 \end{bmatrix} \qquad \text{(by applying } R_3 \rightarrow R_3 - 5R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -25 & -8 \end{bmatrix}$$

which is triangular matrix.

Example 3 : Show that $A = \begin{bmatrix} 3 & 4 & -5 \\ 1 & 1 & 0 \\ 1 & 1 & 5 \end{bmatrix}$ is row equivalent to I_3.

Solution :

$$A = \begin{bmatrix} 3 & 4 & -5 \\ 1 & 1 & 0 \\ 1 & 1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & -5 \\ 1 & 1 & 5 \end{bmatrix} \qquad (R_1 \leftrightarrow R_2)$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 5 \end{bmatrix} \text{ (by } R_2 \rightarrow R_2 \rightarrow 3R_1 \text{ and } R_3 \rightarrow R_3 \rightarrow R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -5 \\ 0 & 0 & 5 \end{bmatrix} \text{ (by } R_1 \rightarrow R_1 - R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -5 \\ 0 & 0 & 5 \end{bmatrix} \text{ (by } R_3 \rightarrow \tfrac{1}{5} R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ (by } R_1 \rightarrow R_1 - 5R_3 \text{ and } R_2 \rightarrow R_2 + 5R_3\text{)}$$

$$= I_3$$

In above example, we have reduced the square matrix A to identity matrix by elementary row operations. Can every square matrix be reduced to identity matrix by elementary row opearations.

The answer, in general, is no, however, if A is a square matrix with $|A| \neq 0$, then A can be reduced to identity matrix by elementary row operations. This we state below without proof.

Theorem : Every non-singular matrix is row equivalent to a unit matrix.

Below we given an algorithm to reduce a non-singular matrix to identity matrix.

1. Make the first element of first column unity by scaling. If the first element is zero the first make use of interchange.
2. Make all elements of first column below the first element zero by using replacement.
3. Now make the second element of second column unity and all other elements zero.
4. Continue the process column by column to get an identity matrix.

The following example illustrate the process.

Example 4 : Reduce the matrix $\begin{bmatrix} 0 & 3 & -3 \\ 2 & -4 & 8 \\ -1 & 3 & -3 \end{bmatrix}$ to I_3.

Solution :

$$\begin{bmatrix} 0 & 3 & -3 \\ 2 & -4 & 8 \\ -1 & 3 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -4 & 8 \\ 0 & 3 & -3 \\ -1 & 3 & -3 \end{bmatrix} \qquad (\text{by } R_1 \leftrightarrow R_2)$$

$$\sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -3 \\ -1 & 3 & -3 \end{bmatrix} \qquad (\text{by } (R_1 \leftrightarrow \tfrac{1}{2} R_1)$$

$$\sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -3 \\ 0 & 1 & 1 \end{bmatrix} \qquad (\text{by } R_3 \rightarrow R_3 + R_1)$$

$$\sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \qquad (\text{by } R_2 \rightarrow \tfrac{1}{3} R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \quad (\text{by } R_1 \rightarrow R_1 + 2R_2 \text{ and } R_3 \rightarrow R_3 - R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{by and } R_3 \rightarrow \tfrac{1}{2} R_3)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{by } R_1 \rightarrow R_1 - 2R_3 \text{ and } R_2 \rightarrow R_2 + R_3)$$

Check Your Progress – 1

1. Write the Matrices obtained by applying the following elementary row operations on

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

(i) $R_1 \leftrightarrow R_3$

(ii) $R_2 \to R_2 + 3R_1$

(iii) $R_2 \to R_{3,}$ then $R_2 \to 2R_2$ and then $R_3 \to R_{3+} 2R_1$

2. Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 3 & 8 \\ 6 & 7 & 2 \end{bmatrix}$ to triangular form.

3. Show that $\begin{bmatrix} 2 & 4 & 6 \\ 3 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$ is row equivalent to $I_{3.}$

4. Is the matrix $\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$ row equivalent to $I_{3.}$

5. Which of the following is row equivalent to $I_{3.}$

(a) $\begin{bmatrix} 0 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 1 \\ -1 & 6 & 2 \end{bmatrix}$

3.3 RANK OF A MATRIX

Suppose A is an $m \times n$ matrix. We can obtain square sub matrices of order r ($0 < r \leq$ least of m and n) from A by selecting the elements in any r rows and r columns of A. We define rank of matrix as follows :

Defintion : Let A be an $m \times n$ matrix. The order of the largest square submatrix of A whose determinant has a non-zero value is called the **'rank'** of the matrix A. The rank of the zero matrix is defiend to be zero.

It is clear from the definition that the rank of a square matrix is r if and only if A has a square submatrix of order r with nonzero determinant, and all square sub matrices of large size have determinant zero.

Example 5 : Find the rank of the matrix.

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 3 & 1 & 2 \\ 2 & 4 & 6 \end{bmatrix}$$

Solution : Since A is a square matrix, A is itself a square submatrix of A.

Also, $|A| = \begin{vmatrix} 0 & 1 & -1 \\ 3 & 1 & 2 \\ 2 & 4 & 6 \end{vmatrix}$

$= -1(18-4) + (-1)(12-2)$

$= -24 \neq 0$

Hence, rank of A is 3.

Example 6 : Determine the rank of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 8 \end{bmatrix}$$

Solution : Here, $|A| = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 8 \end{bmatrix}$

$= 0$

So, rank of A cannot be 3.

Now $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is a square submatrix of A such that $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$

$\therefore$ rank of A = 2.

Rank and Elementary Operations

The following theorem gives a relationship between rank of a matrix and elementary row operations on the matrix.

Theorem : The rank of a matrix remains unaltered under elementary row operations.

The theorem can be proved by noting that the order of the largest non-singular square submatrix of the matrix is not affected by the elementary row operations. Using properties of determinants, we can see that interchange will only change the sign of determinants of square submatrices, while under scaling values of determinants are multiplied by non zero constant and replacement will not affect the value of the determinant.

Using the above theorem, we can obtain the rank of a matrix A by reducing it to some simpler form, say triangular form or normal form.

Example 7 : Determine the rank of matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \\ 5 & 3 & 14 & 4 \end{bmatrix}$$

Solution : We first reduce matrix A to triangular form by elementary row operations.

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 0 \\ 5 & 3 & 14 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & -1 & 2 & 1 \\ 1 & 1 & 2 & 0 \\ 5 & 3 & 14 & 4 \end{bmatrix} \qquad \text{(by } R_1 \leftrightarrow R_2\text{)}$$

$$\sim \begin{bmatrix} 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 8 & 4 & 4 \end{bmatrix} \qquad \text{(by } R_3 \rightarrow R_3 - 5R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & 4 \end{bmatrix} \qquad \text{(by } R_3 \rightarrow R_3 - 8R_2\text{)}$$

We have thus reduced A to triangular form. The reduced matrix has a square

submatrix $\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{bmatrix}$ with non zero

determinant $\begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{vmatrix} = 1\times1\times(-12) = -12.$

So rank of reduced matrix is 3. Hence rank of A = 3.

Example 8 : Reduce the matrix

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 \\ 4 & 7 & -4 & -3 \\ 6 & 9 & -5 & 2 \\ 0 & -9 & 6 & 5 \end{bmatrix}$$

to triangular form and hence determine its rank.

Solution : Let us first reduce A to triangular form by using elementary row operations.

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 \\ 4 & 7 & -4 & -3 \\ 6 & 9 & -5 & 2 \\ 0 & -9 & 6 & 15 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 5 & -3 & -4 \\ 0 & -3 & 2 & 5 \\ 0 & -6 & 4 & 14 \\ 0 & -9 & 6 & 15 \end{bmatrix} \quad \text{(by } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1\text{)}$$

$$\sim \begin{bmatrix} 2 & 5 & -3 & -4 \\ 0 & -3 & 2 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(by } R_3 \to R_3 - 2R_2,\ R_4 \to R_4 - 3R_2\text{)}$$

$$= B$$

Clearly, rank of B cannot B 4; as $|B| = 0$.

Also, $\begin{bmatrix} 2 & 5 & -4 \\ 0 & -3 & 5 \\ 0 & 0 & 4 \end{bmatrix}$ is a square submatrix

of order 3 of B and $\begin{vmatrix} 2 & 5 & -4 \\ 0 & -3 & 5 \\ 0 & 0 & 4 \end{vmatrix} = 2 \times (-3) \times 4 = -24 \neq 0$

So, rank of matrix B is 3.

Hence rank of matrix A = 3.

Normal form of a Matrix

We can find rank of a matrix by reducing it to normal form.

Definition : An $m \times n$ matrix of rank r is said to be in normal form if it is of type.

$$\begin{bmatrix} I_r & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{bmatrix}$$

For example, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is the normal form $\begin{bmatrix} I_2 & O_{2,2} \\ O_{1,2} & O_{1,2} \end{bmatrix}$. We can also

write it as $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$.

Similarly $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ is the normal form $[I_3 \quad 0]$.

In section 1, we discussed elementary row operations. We can similarly define elementary column operations also. An elementary operations is either an elementary row operation or an elementary column operation. A matrix A is equivalent to matrix B if B can be obtained from A by a sequence of elementary operations.

Theorem : Every matrix can be reduced to normal form by elementary operations.

We illustrate the above theorem by following example.

Example 9 : Reduce the matrix

$$A = \begin{bmatrix} 5 & 3 & 8 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

to normal form by elementary operations.

Solution : $A = \begin{bmatrix} 5 & 3 & 8 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_3$, we have

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 5 & 3 & 8 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - 5R_1$, we have

$$A \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 8 & 8 \end{bmatrix}$$

Applying elementary row operations $R_1 \rightarrow R_1 + R_2$ and $R_3 \rightarrow R_3 - 8R_2$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, we apply elementary column operation $C_3 \rightarrow C_3 - C_2$, to get

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Again, applying $C_3 \rightarrow C_3 - C_1$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have thus reduced A to normal form.

Also, note that the rank of a matrix remains unaltered under elementary operations.

Thus, rank of A in above example is 2 because rank of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 2.

In this regard, we state following theorem without proof.

Theorum : Every matrix of rank r is equivalent to the matrix $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

Example 10 Reduce the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 3 & 1 & 3 \end{bmatrix}$$ to its normal form and hence determine its rank.

Solution : We have

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 3 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -1 & 2 \end{bmatrix} \text{ [by } R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 2R_1]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -1 & 2 \end{bmatrix} \text{ [by } C_3 \to C_3 - 2C_1, \quad C_4 \to C_4 - C_1]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} \text{ [by } R_3 \to R_3 - 3R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ [by } R_3 \to \tfrac{1}{2}R_3]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ [by } C_3 \to C_3 + C_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ [by } C_4 \to C_4 - C_3]$$

Thus, A is reduced to normal form $[I_3\ 0]$ and hence rank of A is 3.

Check Your Progress – 2

1 By finding a non-zero minor of largest order determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

2 Reduce the matrix $A = \begin{bmatrix} 0 & 2 & 4 & 6 \\ 3 & -1 & 4 & -2 \\ 6 & -1 & 10 & -1 \end{bmatrix}$ to triangular form and hence determine its rank.

3. Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & -3 & 3 \end{bmatrix}$$

by reducing to triangular form.

4. Reduce the matrix $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 3 & 1 & 1 & 3 \end{bmatrix}$ to its normal form and hence determine its rank.

5. Reduce the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

to its normal form and hence determine its rank.

3.4 INVERSE OF A MATRIX USING ELEMENTARY ROW OPERATIONS

In this section, we shall disucss a method of finding inverse of a square matrix using elementary row operations. We begin by stating the following theorem (without proof) which we require for our discussion.

Theorem : An elementary row operation on the product of two matrices is equivalent to the same elementary row operation on the pre-factor of the product. Recall that an invertible matrix is non singular and that every non singular matrix can be reduced to an identity matrix using elementary row operations only. We now disucss a method of computing inverse of a square matrix using elementary row operations. Let A be an $n\times n$ matrix whose inverse is to be found. Consider the identity $A = I_nA$ where I_n is the identity matrix of order n. Reduce the matrix

A on the L.H.S. to the identity matrix I_n by elementary row operation. Note that this is possible if A is invertible (i.e., non-singular). Now apply all these operations (in the same order) to the pre-factor I_n M the R. H. S. of the identity. In this way, the matrix I_n reduced to some matrix B such that BA = I_n. The matrix B so obtained is the inverse of A.

We illustrate the method in the following examples.

Example 11 : Find the inverse of the matrix.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

using elementary row operations.

Solution : Consider the identity.

$$A = I_2 A$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A \qquad [\text{ by applying } R_2 \to R_2 - 2R_1]$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} A \qquad [\text{ by applying } R_1 \to R_1 - R_2]$$

that is, I_2 = BA

Where $B = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

Thus $A^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

Example 12 : Using elementary row operations find the inverse of the matrix.

$$A = \begin{bmatrix} 2 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Solution : Consider

$$A = I_3 \ A$$

$$\begin{bmatrix} 2 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix}1 & 2 & 3\\2 & 5 & 3\\1 & 0 & 8\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix} A \text{ [by } R_1 \rightarrow \quad R_2]$$

$$\Rightarrow \begin{bmatrix}1 & 2 & 3\\0 & 1 & -3\\0 & -2 & 5\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\1 & -2 & 0\\0 & -1 & 1\end{bmatrix} A \text{ [by } R_2 \rightarrow R_2 - 2R_1,\ R_3 \rightarrow R_3 - R_1]$$

$$\Rightarrow \begin{bmatrix}1 & 0 & 9\\0 & 1 & -3\\0 & 0 & -1\end{bmatrix} = \begin{bmatrix}-2 & 5 & 0\\1 & -2 & 0\\2 & -5 & 1\end{bmatrix} A \text{ [by } R_1 \rightarrow R_1 - 2R_2,\ R_3 \rightarrow R_3 - 2R_2]$$

$$\Rightarrow \begin{bmatrix}1 & 0 & 9\\0 & 1 & -3\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-2 & 5 & 0\\1 & -2 & 0\\-2 & 5 & -1\end{bmatrix} A \text{ [by } R_3 \rightarrow (-1)R_3]$$

$$\Rightarrow \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A = \begin{bmatrix}16 & -40 & 9\\-5 & 13 & -3\\-2 & -5 & -1\end{bmatrix} A \text{ [by } R_1 \rightarrow R_1 - 9R_3,\ R_2 \rightarrow R_2 + 3R_3]$$

that is, $I_3 = BA$

Where $B = \begin{bmatrix}16 & -40 & 9\\-5 & 13 & -3\\-2 & 5 & -1\end{bmatrix} A$

Hence $A^{-1} = \begin{bmatrix}16 & -40 & 9\\-5 & 13 & -3\\-2 & 5 & -1\end{bmatrix}$

Example 13 : Find the inverse, if exists, of the matrix.

$$A = \begin{bmatrix}0 & 1 & 2\\1 & 2 & 3\\3 & 1 & 1\end{bmatrix}$$

Solution : Consider

$$A = I_3A$$

$$\begin{bmatrix}0 & 1 & 2\\1 & 2 & 3\\3 & 1 & 1\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\3 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A \text{ [by } R_1 \leftrightarrow R_2]$$

$$\Rightarrow \begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\3 & -5 & -8\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & -3 & 1\end{bmatrix} A \text{ [by } R_3 \rightarrow R_3 - 3R_1]$$

$$\Rightarrow \begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 2\end{bmatrix} = \begin{bmatrix}-2 & 1 & 0\\1 & 0 & 0\\5 & -3 & 1\end{bmatrix} A \text{ [by } R_1 \to R_1 - 2R_2,\ R_3 \to R_3 + 5R_2]$$

$$\Rightarrow \begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-2 & 1 & 0\\1 & 0 & 0\\5/2 & -3/2 & 1/2\end{bmatrix} A \text{ [by } R_3 \to \tfrac{1}{2}R_3]$$

$$\Rightarrow \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1/2 & -1/2 & 1/2\\-4 & 3 & -1\\5/2 & -3/2 & 1/2\end{bmatrix} A \text{ [by } R_1 \to R_1 + R_3,\ R_2 \to R_2 - 2R_3]$$

that is, $I_3 = \begin{bmatrix}1/2 & -1/2 & 1/2\\-4 & 3 & -1\\5/2 & -3/2 & 1/2\end{bmatrix} A$

Hence $A^{-1} = \begin{bmatrix}1/2 & -1/2 & 1/2\\-4 & 3 & -1\\5/2 & -3/2 & 1/2\end{bmatrix}$

Example 14 : Find the inverse of A, if it exists, for the matrix

$$A = \begin{bmatrix}1 & 6 & 4\\2 & 4 & -1\\-1 & 2 & 5\end{bmatrix}$$

Solution : Consider the identity

$$A = I_3 A$$

$$\begin{bmatrix}1 & 6 & 4\\2 & 4 & -1\\-1 & 2 & 5\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A$$

Again, applying $R_2 \to R_2 - 2R_1$ and , $R_3 \to R_3 + R_1$, we get

$$\begin{bmatrix}1 & 6 & 4\\0 & -8 & -9\\0 & 8 & 9\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\1 & 0 & 1\end{bmatrix} A$$

Again, applying $R_3 \to R_3 + R_2$, we have

$$\begin{bmatrix}1 & 6 & 4\\0 & -8 & -9\\0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\1 & 0 & 1\end{bmatrix} A$$

Since, we have obtained a row of zeros on the L.H.S., we see that A cannot be reduced to an identity matrix. Thus, A is not invertible, Infact, note that A is a singular matrix as |A| = 0.

Check Your Progress – 3

1. Find the inverse of the following Matrices using elementary row operations only.

   (a) A = $\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$

   (b) A = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$

2. Using elementary row operations find the inverse of the matrix

   A = $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

3. Let A = $\begin{bmatrix} 2 & 4 & 6 \\ 3 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$. Find A^{-1} if exists.

4. Find the inverse of matrix A, if it exists, where A = $\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$.

3.5 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. (i) $\begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 8 \\ 1 & 4 & 7 \end{bmatrix}$

   (ii) $\begin{bmatrix} 1 & 4 & 7 \\ 5 & 17 & 29 \\ 3 & 6 & 9 \end{bmatrix}$

   (iii) A ~ $\begin{bmatrix} 1 & 4 & 7 \\ 3 & 6 & 9 \\ 2 & 5 & 8 \end{bmatrix}$ (by $R_2 \leftrightarrow R_3$)

   ~ $\begin{bmatrix} 1 & 4 & 7 \\ 6 & 12 & 18 \\ 2 & 5 & 8 \end{bmatrix}$ (by $R_2 \rightarrow 2R_2$)

   ~ $\begin{bmatrix} 1 & 4 & 7 \\ 6 & 12 & 18 \\ 4 & 13 & 22 \end{bmatrix}$ (by $R_3 \rightarrow R_3 \rightarrow 2R_1$)

2. A = $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 3 & 8 \\ 6 & 7 & 2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -7 \\ 0 & -5 & -16 \end{bmatrix} (\text{ by } R_2 \rightarrow R_2 \rightarrow 5R_1 \text{ and } R_3 \rightarrow R_3 - 6R_1)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -16 \end{bmatrix} (\text{ by } R_2 \rightarrow \frac{-1}{7} R_2)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -11 \end{bmatrix} (\text{ by } R_3 \rightarrow R_3 + 5R_2)$$

which is triangular matrix.

Note : There are many other ways and solutions also using different sequence of elementary row operations.

3. Let $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

Then $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad (\text{ by } R_1 \rightarrow \frac{1}{2} R_1)$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 1 & -1 \end{bmatrix} \quad (\text{ by } R_2 \rightarrow R_2 - 3R_1)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & -5 & -7 \end{bmatrix} \quad (\text{ by } R_2 \leftrightarrow R_3)$$

$$\sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & -12 \end{bmatrix} \quad (\text{ by } R_1 \rightarrow R_1 \rightarrow 2 R_2 \text{ and } R_3 \rightarrow R_3 \rightarrow 5 R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{ by } R_3 \rightarrow -\frac{1}{12} R_3)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{ by } R_1 \rightarrow R_1 - 5 R_3 \text{ and } R_2 \rightarrow R_2 + R_3)$$

$= I_3$.

4. $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & 3 \end{bmatrix}$

So, $|A| = 1(6 + 2) - 2(-3-5) - 3(-2 + 10)$

$= 8 + 16 - 24 = 0$

So, A is a singular matrix. Hence A is not row equivalent to I_3.

5. (a) Let A = $\begin{bmatrix} 0 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

Then, $|A| = -2(2-0) + 3(0-12)$

$= -40 \neq 0$

So, A is non-singular and hence row-equivalent to I_3.

(b) Let A = $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 1 \\ -1 & 6 & 2 \end{bmatrix}$

Then, $|A| = -2(6-6) + 0 + 0 = 0$

$\therefore$ A is not row-equivalent to I_3.

Check Your Progress – 2

1. Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

So $|A| = 1(1-6) - 2(3-4) + 3(9-2)$

$= -6 + 2 + 21$

$= -17 \neq 0$

So, |A| is largest non zero minor & hence rank A = 3

2. A $\sim \begin{bmatrix} 3 & -1 & 4 & -2 \\ 0 & 2 & 4 & 6 \\ 6 & -1 & 10 & -1 \end{bmatrix}$ (by $R_1 \leftrightarrow R_2$)

$\sim \begin{bmatrix} 3 & -1 & 4 & -2 \\ 0 & 2 & 4 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ (by $R_3 \to R_3 - 2R_1$)

$\sim \begin{bmatrix} 3 & -1 & 4 & -2 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (by $R_3 \to R_3 - \frac{1}{2}R_2$)

Where B is a traingular matrix. Clearly every 3– rowed minor of B has value zero.

Also since $\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} = 6 \neq 0$, so rank of B = 2. Hence rank of A = 2.

3. By applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$ and $R_4 \to R_4 - 3R_1$, we have

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix}$$

Again, applying $R_4 \to R_4 \frac{-6}{8} R_3$ we have

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

Where B is a triangular matrix. Since last row of B consist of zeroes only, therefore rank of B cannot be 4.

Also $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & -8 \end{bmatrix} = 1 \times 2 \times (-8) = -16 \neq 0$

$\therefore$ rank of B = 3 and hence, rank of A = 3.

4. $A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{bmatrix}$ (by $R_1 \leftrightarrow R_2$)

$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{bmatrix}$ (by $R_3 \to R_3 - 3R_1$)

$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ (by $R_1 \to R_1 - 2R_2$, by $R_3 \to R_3 + 5R_2$)

$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (by $R_3 - \frac{1}{2} R_3$)

$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (by $R_1 \to R_1 + R_3$, $R_2 \to R_2 - 2R_3$)

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (by $C_4 \to C_4 - C_1$)

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad (\text{by } C_4 \rightarrow C_4 + C_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad (\text{by } C_4 \rightarrow C_4 - C_3)$$

So normal form of A is $[I_3 \; 0]$. Hence rank of A = 3.

5. $$A \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} \qquad (\text{by } R_2 \rightarrow R_2 - 4R_1)$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 1 & 4 \end{bmatrix} \qquad (\text{by } R_2 \leftrightarrow R_4)$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -8 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad (\text{by } R_1 \rightarrow R_1 + R_2,\; R_3 \rightarrow R_3 - 5R_2,\; R_4 \rightarrow R_4 - 3R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -8 & 4 \end{bmatrix} \qquad (\text{by } R_3 \leftrightarrow R_4)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -12 \end{bmatrix} \qquad (\text{by } R_1 \rightarrow R_1 - 2R_3,\; R_4 \rightarrow R_4 - 8R_3)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -12 \end{bmatrix} \qquad (\text{by } R_4 \rightarrow \frac{-1}{12} R_4)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (\text{by } R_1 \rightarrow R_1 + 3R_4,\; R_2 \rightarrow R_2 - 2R_4,\; R_3 \rightarrow R_3 + 2R_4)$$

$$= I_4 \qquad (\text{by } R_4 \rightarrow \frac{1}{12} R_4)$$

So normal form of A is I_4 and hence rank of A = 4.

Check Your Progress – 3

1 (a) $A = I_2\, A$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}A$$

$$\Rightarrow \begin{bmatrix}1 & 2\\0 & -2\end{bmatrix} = \begin{bmatrix}1 & 0\\-3 & 1\end{bmatrix}A \qquad (\text{by } R_2 \to R_2 - 3R_1)$$

$$\Rightarrow \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\3/2 & -1/2\end{bmatrix} \qquad (\text{by } R_2 \to \frac{-1}{2} R_2)$$

$$\Rightarrow \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}-2 & 1\\3/2 & -1/2\end{bmatrix} \qquad (\text{by } R_1 \to R_1 - 2R_2)$$

Hence $A^{-1}\begin{bmatrix}-2 & 1\\3/2 & -1/2\end{bmatrix}$

(b) $A = I_3\, A$ $\qquad (\text{by } R_2 \to \frac{-1}{2} R_2)$

$$\begin{bmatrix}1 & 1 & 1\\0 & 2 & 3\\5 & 5 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}A$$

$$\Rightarrow \begin{bmatrix}1 & 1 & 1\\0 & 2 & 3\\0 & 0 & -4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-5 & 0 & 1\end{bmatrix}A \qquad (\text{by } R_3 \to R_3 - 5R_1)$$

$$\Rightarrow \begin{bmatrix}1 & 1 & 1\\0 & 1 & 3/2\\0 & 0 & -4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1/2 & 0\\-5 & 0 & 1\end{bmatrix}A \qquad (\text{by } R_2 \to \frac{-1}{2} R_2)$$

$$\Rightarrow \begin{bmatrix}1 & 0 & -1/2\\0 & 1 & 3/2\\0 & 0 & -4\end{bmatrix} = \begin{bmatrix}1 & -1/2 & 0\\0 & 1/2 & 0\\-5 & 0 & 1\end{bmatrix}A \qquad (\text{by } R_1 \to R_1 - R_2)$$

$$\Rightarrow \begin{bmatrix}1 & 0 & -1/2\\0 & 1 & 3/2\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1/2 & 0\\0 & 1/2 & 0\\5/4 & 0 & -1/4\end{bmatrix}A \qquad \left(\text{by } R_3 \to \frac{1}{4} R_3\right)$$

$$\Rightarrow \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}13/8 & -1/2 & -1/8\\-15/8 & 1/2 & 3/8\\5/4 & 0 & -1/4\end{bmatrix}A \qquad \left(\text{by } R_1 \to R_1 + \frac{1}{2} R_3, R_2 \to \frac{-3}{2} R_3\right)$$

Hence, $A^{-1}\begin{bmatrix}13/8 & -1/2 & -1/8\\-15/8 & 1/2 & 3/8\\5/4 & 0 & -1/4\end{bmatrix}$

2. A = I_3 A

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 4 & -3 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad (\text{ by } R_1 \leftrightarrow R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & -4 & 1 \end{bmatrix} A \qquad (\text{ by } R_3 \rightarrow R_3 - 4R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & -4 & 1 \end{bmatrix} A \qquad (\text{ by } R_3 \rightarrow R_3 + 3R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 3/2 & -2 & 1/2 \end{bmatrix} A \qquad \left(\text{by } R_3 \rightarrow \frac{1}{2} R_3\right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} A \quad (\text{by } R_1 \rightarrow R_1 - 3R_3,\ R_2 \rightarrow R_2 - 2R_3)$$

3. A = I_3 A

$$\begin{bmatrix} 2 & 4 & 6 \\ 3 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ -3/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & -5 & -7 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 1 \\ -3/2 & 1 & 0 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & -2 \\ 0 & 0 & 1 \\ -3/2 & 1 & 5 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & -2 \\ 0 & 0 & 1 \\ -3/24 & -1/12 & -5/12 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3/24 & 5/12 & 1/12 \\ 3/24 & -1/12 & 7/12 \\ 3/24 & -1/12 & -5/12 \end{bmatrix} A$$

$$\text{Hence } A^{-1} = \begin{bmatrix} -3/24 & 5/12 & 1/12 \\ 3/24 & -1/12 & 7/12 \\ 3/24 & -1/12 & -5/12 \end{bmatrix}$$

4. $A = I_3 A$

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & -12 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} A \text{ (by } R_2 \to R_2 - R_1, R_3 \to R_3 - 5R_1)$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix} A \quad \text{(by } R_3 \to R_3 - 3R_2)$$

The matrix on L.H.S. has a row of all zeroes. So the matrix A cannot be reduced to an identity matrix. Hence, A is not invertible. Infact note that A is singular as $|A| = 0$.

3.6 SUMMARY

This unit deals with advanced topics on matrices. First of all, in **section 3.2**, the concept of an elementary row operation of a matrix, is given. Then, through examples, it is illustrated how a matrix may be reduced to some standard forms like triangular matrix and identity matrix. In **section 3.3**, a very important concept of rank of matrix, is defined. Through a number of examples, it is explained how rank of a matrix can be found using elementary operations. In **section 3.4**, inverse of an invertible matrix is defined. Finally, through a number of suitable examples, it is explained how inverse of an invertible matrix can be found using elementary operations.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 3.5**.