UNIT 4 INEQUALITIES

Structure

4.0 INTRODUCTION

We are familiar with inequality symbols $>$, $\geq$, $<$ and $\leq$. We have used these symbols to compare two real numbers and subsets of real numbers. For instance, recall that the inequality $x \geq 3$ denote the set of all real numbers x that are greater than or equal to 3. In this unit, we shall extend our work to include more complex inequalities such as :

$$5x - 3 > 2x + 7$$

4.1 OBJECTIVES

After going through this unit, you will be able to:

- find the solution set of a linear inequality and draw its graph on number line;
- find the solution set of a compound inequality;
- find the solution set of an inequality involving absolute value;
- graph solution set of linear inequality in two variables;
- graph solution set of system of simultaneous inequality in two variables; and
- find solution set of quadratic and non-linear inequalities.

4.2 SOLVING A LINEAR INEQUALITY

A linear inequality contains a linear expression on one side of the inequality and a linear expression or a constant on the other side.

For instance

$$2x + 5 > 3 - 7x;$$

$$3x + 7 \geq 8; \text{ and}$$

$$2x - 5 < 6 + 4x$$

are linear inequalities.

By solving a linear inequality in one variable, we mean finding all the values of the variable for which the inequality becomes true.

We first list some tools for solving inequalities.

Tools for Solving Inequalities

The solution set of an inequality will not change if we perform any of the following:

Addition

Add (or subtract) the same number or the both sides of the inequality.

Multiplication by Positive Number

Multiply (or divide) both sides by the same *positive* number.

Multiplication by Negative Number

Multiply (Or divide) both sides by the *negative* number and *reverse* the direction of the inequality symbol.

Solved Examples

Example 1: Solve the inequality

$$x + \frac{2}{3} \geq \frac{7}{2}$$

and graph its solution set.

Solution : The given in equalities is

$$x + \frac{2}{3} \geq \frac{7}{2}$$

$$\Leftrightarrow x + \frac{2}{3} + \left(-\frac{2}{3}\right) \geq \frac{7}{2} + \left(-\frac{2}{3}\right) \qquad \left[add \left(-\frac{2}{3}\right) \text{ to both the sides}\right]$$

$$\Leftrightarrow x \geq \frac{7}{2} - \frac{2}{3}$$

$$\Leftrightarrow x \geq \frac{17}{6}$$

The Solution set is { x | $x \geq 17/6$} = [17/6, ∞]

The graph of this set is

17/6

Example 2: Solve the inequality

$4\,x < 28$

and graph its solution set.

Solution : The given inequality is

$$4\,x < 28$$

$$\Leftrightarrow \quad \frac{1}{4}(4x) < \frac{1}{4}(28) \qquad \text{[Multiply both the sides by 1/4]}$$

$$\Leftrightarrow \quad x < 7$$

CAUTION	As 4 is positive, the direction of the inequality is not reversed.

The solution set is $\{\, x \mid x < 7 \,\} = (-\infty, 7)$

The graph of the set is

Example 3 : Solve the inequality

$$-3\,x \geq \frac{5}{6}$$

Solution : The given inequality is

$$-3\,x \geq \frac{5}{6}$$

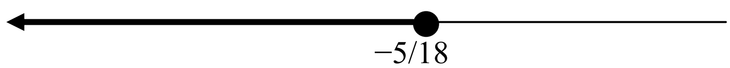
$$\Leftrightarrow \quad \left(-\frac{1}{3}\right)(-3x) \leq \left(-\frac{1}{3}\right)\left(\frac{5}{6}\right)$$

[Multiple both sides by (− 1/3) and reverse the direction of the inequality]

$$\Leftrightarrow \quad x \leq -\frac{5}{18}$$

The solution set is $\{\, x \mid x \leq -5/18 \,\} = (-\infty, -5/18\,]$

The graph of this set is



CAUTION :	Since − 1/3 is a negative number the direction of the inequality must be reversed.

Example 4: Find the solution set of the inequality

$$7\,x + 4 \leq 4\,x + 16$$

Solution : The given inequality

$$7x + 4 \leq 4x + 16$$

We first bring the like terms on one side. This can be achieved by adding $-4x$ to both the sides of the inequality.

$$(-4x) + 7x + 4 \leq (-4x) + 4x + 16$$

$\Leftrightarrow \quad 3x + 4 \leq 16$

$\Leftrightarrow \quad 3x + 4 + (-4) \leq 16 + (-4)$ [add -4 to both the sides]

$\Leftrightarrow \quad 3x \leq 12$

$\Leftrightarrow \quad \frac{1}{3}(3x) \leq \left(\frac{1}{3}\right)(12)$ [Multiply both the sides by 1/3]

$\Leftrightarrow \quad x \leq 4$

Thus, the solution set is $\{x \mid x \leq 4\} = (-\infty, 4]$

Example 5 : Find the solution set of the inequality

$$2(x-1) \geq 7x$$

Solution : The given inequality is

$2(x-1) \geq 7x$

We first remove the parentheses. This can be achieved by using distributive property.

$2x - 2 \geq 7x$

As in the previous problem our next objective is to collect x − terms on one side of the inequality. Towards this end we add $-2x$ to both the sides of the inequality.

$(-2x) + (2x - 2) \geq (-2x) + 7x \Leftrightarrow -2 \geq 5x$

$\Leftrightarrow \quad \frac{1}{5}(-2) \geq \frac{1}{5}(5x)$ [Multiply both sides by 1/5]

$\Leftrightarrow \quad -\frac{2}{5} \geq x.$

Thus, the solution set is $\{x \mid x \leq -2/5\} = (-\infty, 2/5]$

Example 6: Solve the inequality

$-(x-3) + 4 < -2x + 5$

and graph the solution set.

Solution : Remove the parentheses to obtain

$-x + 3 + 4 < -2x + 5$

$\Leftrightarrow 2x - x < 5 - 7$

$\Leftrightarrow x < -2$

$\therefore$ Solution set is $\{x \mid x < -2\} = (-\infty, -2)$

The Graph of the solution set is

Example 7 : Solve the inequality

$$\frac{3}{5}(x - 2) \leq \frac{5}{3}(2 - x)$$

and graph the solution set.

Solution : We first clear fractions by multiply both sides by 15

$$\frac{3(x-2)}{5} \leq \frac{5(2-x)}{3} \Leftrightarrow 3\,[\,3(x - 2] \leq 5[5\,(2-x)]$$

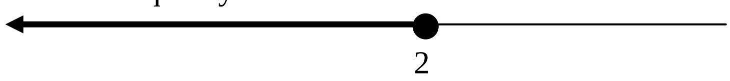
$\Leftrightarrow \quad 9x - 18 \leq 50 - 25x$

$\Leftrightarrow \quad 9x + 25x \leq 50 + 18$

$\Leftrightarrow \quad 34x \leq 68$ or $x \leq 2$

$\therefore$ Solution set is $\{x \mid x \leq 2\} = (-\infty, 2\,]$

The graph of the inequality is



Example 8: Solve the inequality

$$\mathbf{\frac{1}{3}(4 - 2\boldsymbol{x}) \geq \frac{\boldsymbol{x}}{2} - 3}$$

and graph the solution set.

Solution : We first clear fractions by multiplying both sides by 6.

$$6\left(\frac{4-2x}{3}\right) \geq 6\left(\frac{x}{2} - 3\right)$$

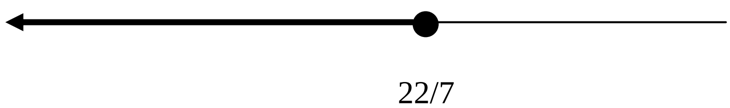
$\Leftrightarrow \quad 2(4 - 2x) \geq 3x - 18$

$\Leftrightarrow \quad 8 - 4x \geq 3x - 18$ [remove paranthesis]

$\Leftrightarrow \quad 8 + 18 \geq 3x + 4x$ or $22 \geq 7x$ or $7x \leq 22$ or $x \leq 22/7$

$\therefore$ Solution set is $\{x \mid x \leq 22/7\} = (-\infty, \frac{22}{7}]$

The graph of the solution set is



Example 9 : Solve the inequality

$$-3 < 4 - 7x < 18$$

and graph the solution set.

Solution : The given inequality

$$-3 < 4 - 7x < 18$$

is equivalent to the inequality

$$-3 - 4 < 7x < 18 - 4 \text{ or } -7 < -7x < 14$$

We divide this compound inequality by − 7 and reverse the inequality signs to obtain

$$\frac{-7}{-7} > \frac{-7x}{-7} > \frac{14}{-7} \text{ or } -2 < x < 1$$

$\therefore$ Solution set is $\{ x \mid -2 < x < 1 \} = (-2, 1)$

The graph of this set is

Example 10 : Solve the inequality

$$-2 < \frac{1}{5}(4 - 3x) \leq 8$$

and graph the solution set.

Solution : We first multiply the given inequality by 5 to obtain

$$(-2)(5) < 4 - 3x \leq (8)(5)$$

$$\Leftrightarrow \quad -10 - 4 < -3x \leq 40 - 4$$

$$\Leftrightarrow \quad \frac{-14}{-3} > x \geq \frac{-44}{4} \quad \Leftrightarrow \quad -11 \leq x < 14/3$$

$\therefore$ Solution set is $\{ x \mid -11 \leq x \ \frac{14}{3}\} = [-11, \frac{14}{3}]$

Graph of the solution set is

Example 11: Solve the inequality

$$\frac{2x - 5}{x + 2} < 2$$

and graph its solution

Solution : The inequality $\frac{2x - 5}{x + 2} < 2$ is

equivalent to

$$\frac{2x + 5}{x + 2} - 2 < 0 \Leftrightarrow \frac{2x - 5 - 2x - 4}{x + 2} < 0$$

$$\Leftrightarrow \frac{-9}{x + 2} < 0$$

But $\frac{a}{b} < 0, a < 0 \Leftrightarrow b > 0.$

Thus, $\frac{-9}{x+2} < 0, \; -9 < 0 \Leftrightarrow x + 2 > 0 \Leftrightarrow x > -2$

$\therefore$ Solution set of the inequality is $(-2, \infty)$

The graph of the solution set is

Example 12 : Solve the inequality

$$\frac{\mathbf{6}}{\boldsymbol{x - 3}} < 5$$

Solution : To solve this inequality, we have to consider two cases.

Case 1: $x < 3$

As $x < 3$, $x - 3 < 0$, therefore $\frac{\mathbf{6}}{\boldsymbol{x-3}} < 0$ and $\frac{\mathbf{6}}{\boldsymbol{x-3}} < 5$

$\therefore$ In this case, solution set of the inequality is $(-\infty, \mathbf{3})$

Case 2: $x > 3$

As $x > 3$, $x - 3 > 0$

$$\therefore \frac{\mathbf{6}}{\boldsymbol{x - 3}} < 5 \, , x - 3 > 0$$

$$\Leftrightarrow \mathbf{6} < 5x - 15$$

$$\Leftrightarrow \mathbf{21} < 5x \Leftrightarrow \boldsymbol{x} > 21/5$$

Thus, in this case solution set of the inequality is

$\{ x \mid x > 21/5 \} = (21/5, \infty)$

Hence, solution of the inequality is $(-\infty, 3) \cup (21/5, \infty)$.

Check Your Progress 1

Solve the following inequalities (1 – 4) and graph the solution set.

1. $2x + 1 > -3$
2. $3x - 2 \geq 4$
3. $2 - 5x \leq 4$
4. $3 + 2x < 7$

Find the solution set of the inequalities (5–8)

5. $3x + 2 \leq 2x + 5$
6. $3(x - 2) \geq 5 - 2x$
7. $-(x - 4) < 2x + 4$
8. $\frac{1}{5}(x - 3) \geq \frac{7}{3}(3 - x)$

Find the solution set of the inequalities (9–10)

9. $-2 \leq 5 - 4x \leq 7$
10. $-3 \leq \frac{1}{4}(x - 2) \leq 4$
11. $\frac{x+3}{x-1} < 0$
12. $\frac{x+8}{x+1} > 1$

4.3 INEQUALITIES AND ABSOLUTE VALUE

Inequalities often occur in combination with absolute value, such as $|x - 7| < 2$ or $|x + 1| > 2$. In order to see how to solve these, we first list some useful properties of absolute value.

We first recall that if $x \in \mathbf{R}$, then

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Note that $|x| = \max\{-x, x\}$.

Properties of Absolute Values

1. $|-x| = |x| \quad \forall x \in \mathbf{R}$
2. $|xy| = |x||y| \quad \forall x, y \in \mathbf{R}$
3. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|} \quad \forall x, y \in \mathbf{R}\ y \neq 0$
4. $|x| = \sqrt{x^2} \quad \forall x \in |\mathbf{R}$
5. If $a > 0$, then

   $|x| \leq a \Leftrightarrow -a \leq x \leq a$

   $|x| \geq a \Leftrightarrow -x \leq a$ or $x \geq a$

6. If $a > 0$, then $|x - y| < a$

$\Leftrightarrow -a < x - y < a \Leftrightarrow y - a < x < y + a$

7. Triangle Inequality

$|x + y| \leq |x| + |y| \quad \forall\, x, y \in \boldsymbol{R}$

The equality holds if and only if $xy > 0$

Solved Examples

Example 13 : Solve the inequality

$$\left|\frac{3x-4}{2}\right| \leq \frac{5}{12} \text{ and graph the solution set.}$$

Solution : We use the property $|y| \leq b \Leftrightarrow -b \leq y \leq b$. Thus

$$\left|\frac{3x-4}{2}\right| \leq \frac{5}{12} \quad \Leftrightarrow \quad -\frac{5}{12} \leq \frac{3x-4}{2} \leq \frac{5}{12}$$

$$\Leftrightarrow 2\left(-\frac{5}{12}\right) \leq 3x - 4 \leq 2\left(\frac{5}{12}\right) \Leftrightarrow \frac{-5}{6} + 4 \leq 3x \leq 4 + \frac{5}{6}$$

$$\Leftrightarrow \frac{-19}{6} \leq 3x \leq \frac{29}{6} \Leftrightarrow \frac{-19}{18} \leq x \leq \frac{29}{18}$$

Thus, solution set of the inequality is $\left[\frac{-19}{18}, \frac{29}{18}\right]$

The graph of this set is

Example 14 : Find solution set of the inequality

$$\left|x + \frac{1}{4}\right| > \frac{7}{4}$$

Solution : We use the property $|y| > b \Leftrightarrow y > b$ *or* $y < -b$. *Thus,*

$$\left|x + \frac{1}{4}\right| > \frac{7}{4} \Leftrightarrow x + \frac{1}{4} > \frac{7}{4} \text{ or } x + \frac{1}{4} \leq -\frac{7}{4}$$

$$\Leftrightarrow x > \frac{7}{4} - \frac{1}{4} \text{ or } x < -\frac{7}{4} - \frac{1}{4} \Leftrightarrow x > \frac{3}{2} \text{ or } x < -2$$

$\therefore$ Solution set of the inequality is

$$\left\{x \,\middle|\, x > \frac{3}{2} \text{ or } x < -2\right\} = (-\infty, -2) \cup \left(\frac{3}{2}, \infty\right)$$

Example 15 : Solve the inequality

$$\frac{2}{|x-3|} > 5 \text{ and graph its solution.}$$

Solution : Note that the domain of the inequality is{ $x \mid x \neq 3$)

For $x \neq 3$, $|x - 3| > 0$

Thus, the given inequality

$$\frac{2}{|x-3|} > 5 \Leftrightarrow 2 > 5|x-3|$$

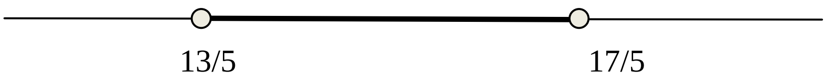
$$\Leftrightarrow |x-3| < \frac{2}{5} \Leftrightarrow \frac{-2}{5} < x - 3 < \frac{2}{5}$$

$$\Leftrightarrow 3 - \frac{2}{5} < x < 3 + \frac{2}{3} \Leftrightarrow \frac{13}{5} < x < \frac{17}{5}$$

$\therefore$ the solution set of the inequality is

$\{x \mid 13/5 < x < 17/5\} = (13/5, 17/5)$

The graph of this set is



Application of inequalities

Inequalities arise in applications just as equations do. We use the following phrases for inequalities.

Statement	**Symbols**
a is more than b	$a > b$
a is less than b	$a < b$
a is not less than b	$a \geq b$
a is not more than b	$a \leq b$
a is at most b	$a \leq b$
a is at least b	$a \geq b$

Example 16 : Translate the following statements into an inequality.

(i) x is a most 7 (ii) – 4 is less than y

(iii) Seven more than z is at least 7 (iv) Twice x is greater than y

(v) The square root of y is at least 7 (vi) The opposite of z is less than y

Solution :

(i) x is a most 7 :

$$x \leq 7$$

(ii) – 4 is less than y

$$-4 < y$$

(iii) Seven more than z (i.e., $7 + z$) is at least 15 :

$$7 + z \geq 15$$

(iv) Twice of x (i.e., $2x$) is greater than y :

$$2x > y$$

(v) The square root of y is (i.e., $\sqrt{y}$) is at least 7 :

$$\sqrt{y} \geq 7$$

(vi) The opposite of z (i.e., $-z$) is less than y

$$-z \leq y$$

Example 17 : Vishi obtained 73, 67 and 72 marks in her first three mathematics tests. How much marks should she get in her fourth test so as to have an average of at least 75 ?

Solution: Let x denote her marks in the fourth test. The sum of marks obtained by her in four tests divided by 4 is her average marks. This, average is to be at least 75. So we have the inequality

$$\frac{73 + 67 + 72 + x}{4} \geq 75 \iff \frac{212 + x}{4} \geq 75$$

$$\iff 4\left(\frac{212 + x}{4}\right) \geq 4\,(75) \iff 212 + x \geq 300$$

$$\iff 212 + x - 212 \geq 300 - 212 \iff x \geq 88$$

Thus, she must get at least 88 marks in her fourth test.

Example 18 : The sum of three consecutive integers be no more than 12. What are the integer.

Solution : Let three consecutive integers be $x - 1$, x and $x + 1$. Sum of these integers is no more than 12 means:

$$(x - 1) + x + (x + 1) \leq 12 \iff 3x \leq 12 \iff x \leq 4$$

Thus, the integers must be $x - 1$, x and $x + 1$ with $x \leq 4$

Check Your Progress – 2

Solve the following inequalities :

1. $| x - 3 | \geq 2$

2. $\left|\frac{2x-5}{3}\right| \leq 1$

3. $\left|\frac{x-5}{3}\right| < 6$

4. $\frac{5}{|x-3|} < 7$

5. In drilling world's deepest hole, it was found that the temperature T in degree Celsius x km below the surface of the earth, was given by the formula

   $T = 30 + 25\ (x - 3),\quad 3 \leq x \leq 15$

   At what depth your expect to find temperature between 200℃ and 300℃.

6. IQ (Intelligence Quotient) of a person is given by the formula

   $\text{IQ} = \frac{MA}{CA} \times 100$

   Where MA stands for the mental age of the person CA for the chronological age. If $80 \leq$ IQ ≤ 140 for a group of 12 year old children, find the range of their mental age.

7. The cost of manufacturing x telephones by Josh Mobiles is given by C = 3000 + 200x, and the revenue from selling these are given by R = 300x. How many telephones must be produced and sold in order to realize a profit ?

   [**Hint :** Profit = revenue − cost]

8. A doctor has prescribed 3 cc of medication for a patient. The tolerance (that is, the amount by which the medication may differ from the acceptable amount) is 0.005 cc. Find the lower and upper limits of medication to be given.

4.4 LINEAR INEQUALITIES IN TWO VARIABLES

A Linear inequality in two variables is an expression that can be put in the form

$ax + by < c$

where a, b and c are real numbers (where a and b are not both 0's). The inequality symbol can be any one of the following four :

$<,\ \leq,\ >,\ \geq$

Some examples of linear inequalities are

$x + 3y < 6$, $y - x > 1$, $x + 2y \geq 1$, $2x + 3y \geq 6$

The solution set for a linear inequality is a section of the coordinate plane. The boundary for the section is found by replacing the inequality symbol with an equal sign and representing the resulting equation in graphical form. The boundary is included in the solution set (and represented with a solid line) if the inequality symbol used originally is $\leq$ or $\geq$ The boundary is not included (and is represented with a broken line) if the original symbol is < or >.

Let us look at some illustrations.

Illustration : Represent graphically the solution set for $2x + 3y \geq 6$.

Solution

The boundary for the graph is the plot of the equation $2x + 3y = 6$. for which the x–intercept and y–intercept are 3 and 2, respectively. (Recall that x–intercept is obtained by letting $y = 0$ and y–intercept by letting $x = 0$). The boundary is included in the solution set, because the inequality symbol is $\geq$.

The graph of the boundary is given in Figure 4.1

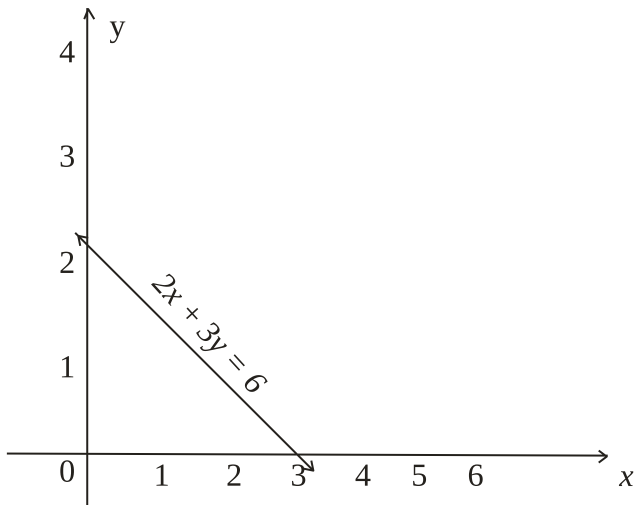


Figure 4.1

The boundary splits the coordinate plane into two sections or regions, the region above the boundary and the region below the boundary. The solution set for $2x + 3y \geq 6$ is one of these two regions, along with boundary. To find the correct region, we simply choose any convenient point this is not on the boundary. We substitute the coordinates of the point in the original inequality $2x + 3y \geq 6$. If the point we choose satisfies the inequality, then it is a member of the solution set and all points on the same side of the boundary as that of the chosen points are also in the solution set. If the coordinates of our point do not satisfy the original inequality, then solution set lies on the other side of the boundary.

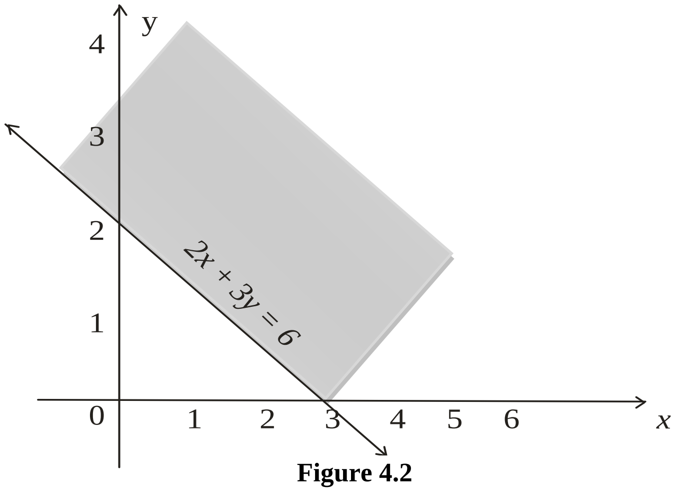


Figure 4.2

In this illustration, a convenient point off the boundary is the origin.

Substituting (0,0)

into $2x + 3y \geq 6$

gives us $2(0) + 3(0) \geq 6$ or $0 \geq 6$ (a false statement)

Since the origin is *not* a solution of the inequality $2x + 3y \geq 6$ and it lies below the boundary, the solution set must lies on the other side of the boundary.

The graph of $2x + 3y \geq 6$ is given by the shaded part of Figure 4.2.

4.5 PROCEDURE TO GRAPH THE SOLUTION SET OF LINEAR INEQUALITIES IN TWO VARIABLES

Step 1 : Replace the inequality with an equal sign. The resulting equation represents the boundary for the solution set.

Step 2: Represent graphically the boundary found is step 1 using the solid line, if the boundary is included in the solution set (that is, if the original inequality symbol was either $\geq$ or $\leq$). Use a *broken line* to draw the graph of the boundary, if it is not included in the solution set (that is, if the original inequality sign was either < or >).

Step 3 Choose any convenient point not on the boundary and substitute the coordinates into the original inequality. (if the origin does not lie on the boundary, it is preferable to use the origin). If the resulting statement is *true,* the graph lies on the same side of the boundary as the chosen point. If the resulting statement is false, the solution set lies on the *opposite side* of the boundary.

Example 19 Draw the graph of the inequality $y < 2x - 3$

Solution

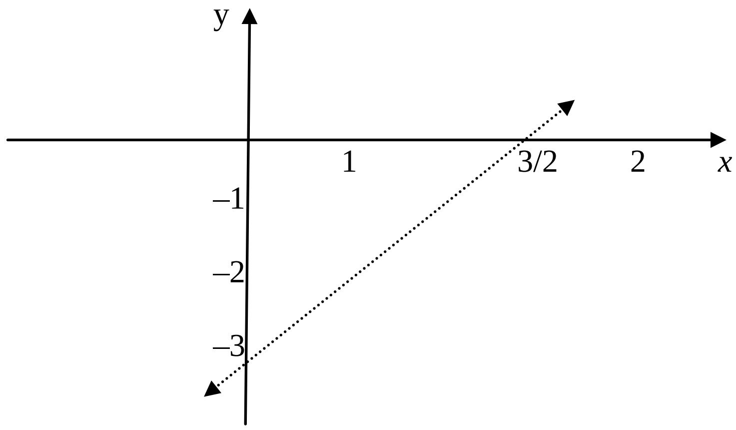


Figure 4.3

Solution

The boundary is the graph of $y = 2x - 3$. It is a line which makes x – intercept of 3/2 and y–intercept of −3. Since the original symbol is <, the boundary is not included in the set. Therefore, we use a broker line to represent the boundary. (See Fig 4.3).

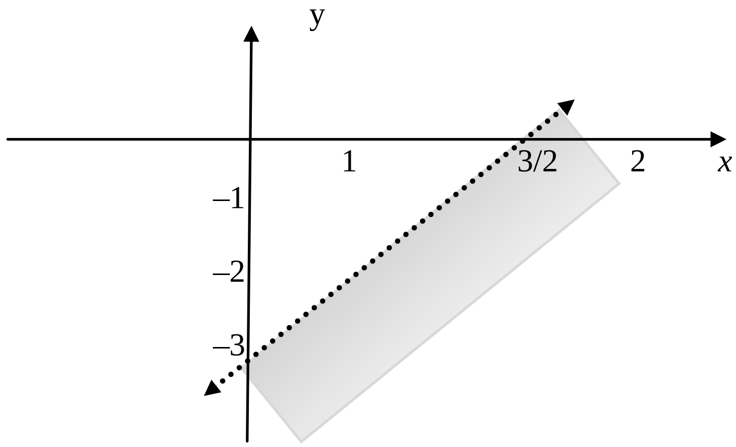


Figure 4.4

We use the origin as a test point. Substituting (0,0) into $y < 2x - 3$, we have $0 < 2(0) - 3$ or $0 < -3$, which is a false statement.

Thus, the origin is not in the solution. Since it lies above the boundary, the solution set lies on the other side of the boundary.

The graph is given in Figure 4.4

Example 20 Draw the graph of the inequality $x + y \le 0$

Solution : The boundary is the graph of $x + y = 0$ It is a line that passes through (0,0) and (−1, 1). Since the boundary passes through the origin, we use (1,0) as a test point. Since (1,0) does not satisfy the inequality $x + y \le 0$ and (1,0) lies above the boundary, the solution set must on the other side of the boundary. (See figure 4.5)

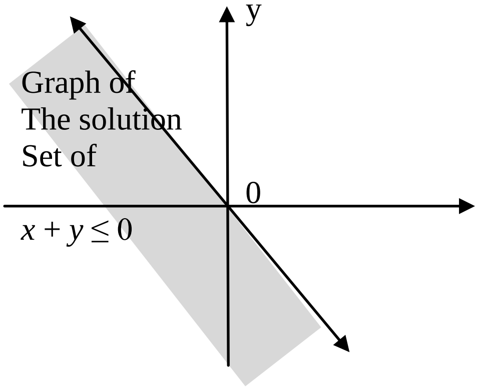


Figure 4.5

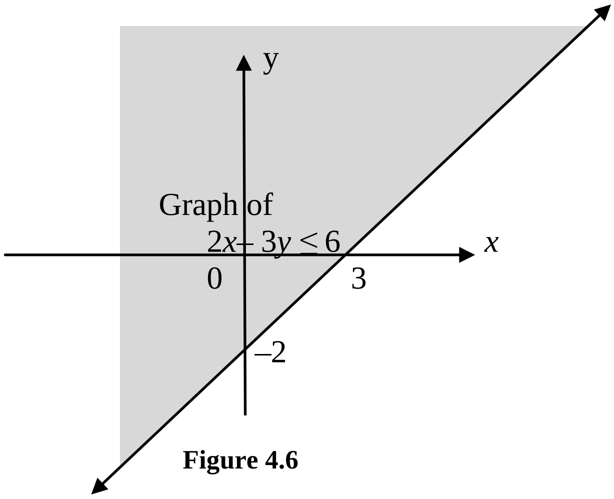


Figure 4.6

Example 21: Draw the graph of the inequality $2x - 3y \le 6$

Solution : The double intercept form of the boundary is $\frac{x}{3} + \frac{y}{-2} = 1$. Test point (0,0) satisfies the inequality. The graph of the inequality is given in Figure 4.6

Example 22: Draw the graph of the following inequalities

(i) $x > 3$ (ii) $y \le 4$

Solution: The boundary is $x = 3$ which is a vertical line. All the points to the right of $x = 3$ have x–coordinate greater than 3 and all the points to the left of $x = 3$ have x coordinate less than 3 (See figure 4.7)

(ii) The boundary is $y = 4$, which is a horizontal line. All the points of the line $y = 4$ and below this line have y coordinate ≤ 4 See Figure 4.8.

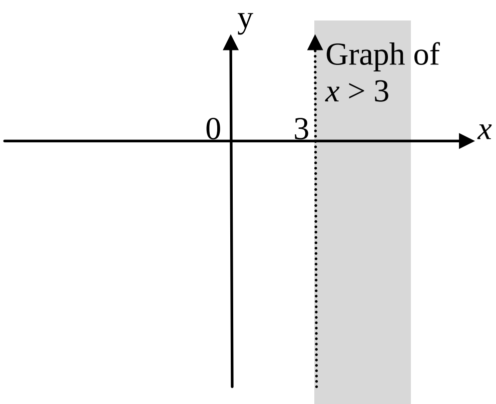


Figure 4.7

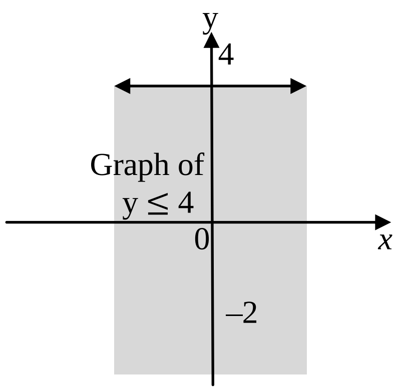


Figure 4.8

4.6 SYSTEM OF LINEAR INEQUALITIES IN TWO VARIABLES

Consider the system of linear inequalities in two variables

$3x + 2y \leq 6$ and $x + 2y \geq 4$

First of all we draw the boundaries $3x + 2y = 6$ (that is, $\left(that\ is, \frac{x}{2} + \frac{y}{3} = 1\right)$ and $x + 2y = 4$ $\left(\text{that is,} \frac{x}{4} + \frac{y}{2} = 1\right)$. We indicate the solution sets of the inequalities by drawing arrows as show in Figure 4.9 and then shade the common (i.e. solution set). See Figure 4.9.

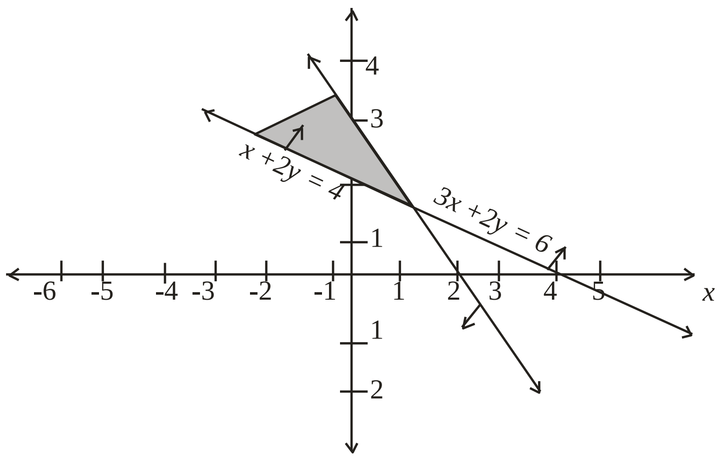


Figure 4.9

Example 23 Draw the graph of the solution set of the following inequalities on the same graph.

$2x + y \geq 8$, $x + 2y \geq 8$ and $x + y \leq 6$.

Solution : We first draw the graphs of the boundaries $2x + y = 8$.

$\left(i.e., \frac{x}{4} + \frac{y}{8} = 1\right), x + 2y = 8 \left(i.e., \frac{x}{8} + \frac{y}{4} = 1\right)$ and $x + y$

$= 6 \left(i.e., \frac{x}{6} + \frac{y}{6} = 1\right)$

We indicate the solution set of each of the inequalities by two arrows and then shade the common region (i.e., shows solution set) in Fig. 4.10.

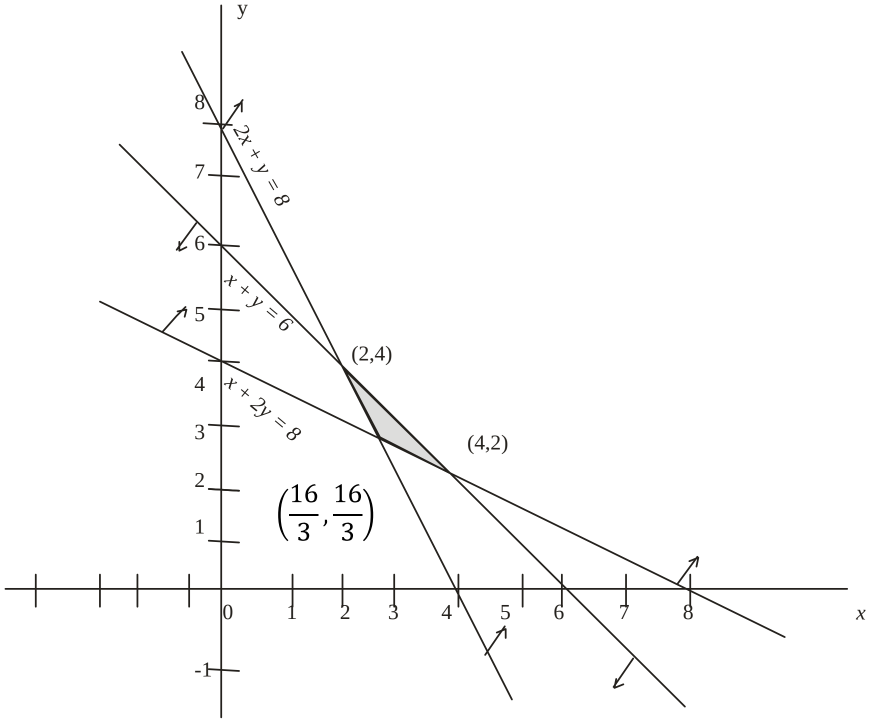


Figure 4.10

Example 24 Draw the solution set of the following inequalities on the same graph.

$$3x - 5y \leq 15,\ \ 3x + 2y \geq 6,\ \ x \leq y \text{ and } x \geq 0$$

Solution : As $x \geq 0$ we shall restrict ourselves to the right of y–axis (including the y–axis). We draw the boundaries $3x - 5y = 15$, $3x + 2y = 6$

$\frac{1}{2}$ and $x = y$. Note that $x = 0$ is y–axis. We next indicate the solution

$\left(i.e., \frac{x}{2} + \frac{y}{3} = 1\right)$ set of each of the inequalities by drawing the arrows as shown in Figure 4.11. We finally shade the common region (i.e., solution set)

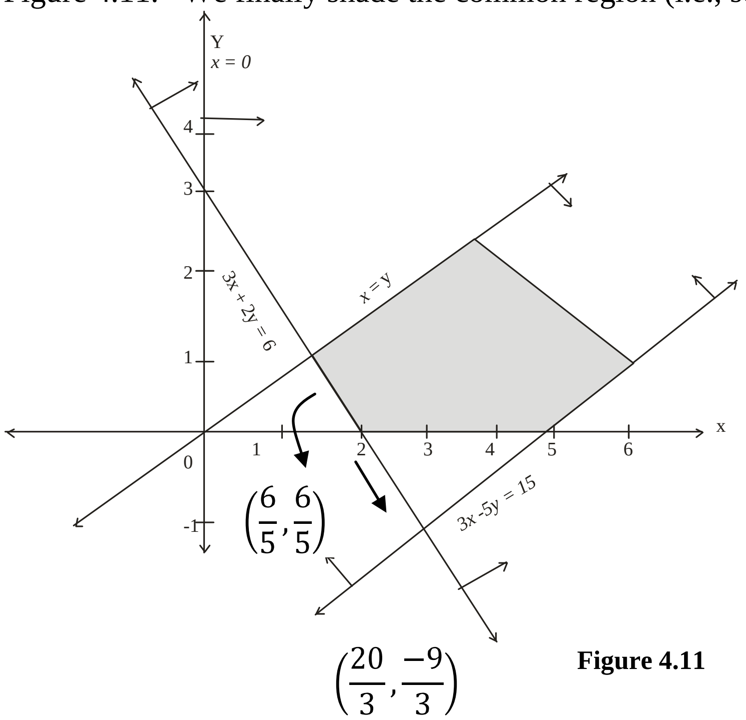


Figure 4.11

Check Your Progress 3

Draw the graph of the following inequalities

1. $x - y \geq -3$
2. $2x + 3y \leq 6$
3. $y < 2x$
4. $x \leq -2$
5. $y > 2$

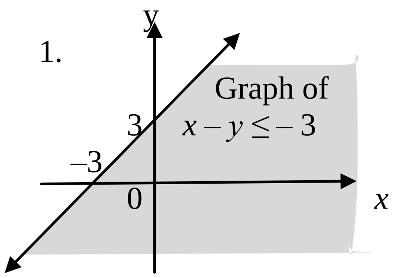
For each of the question 6 to 7 draw the graph of the following inequalities on the same graph.

6. $2x + y \geq 10, \quad x + 4y \geq 12, \quad 6x + y \geq 18, \quad x \geq 0, \quad y \geq 0$

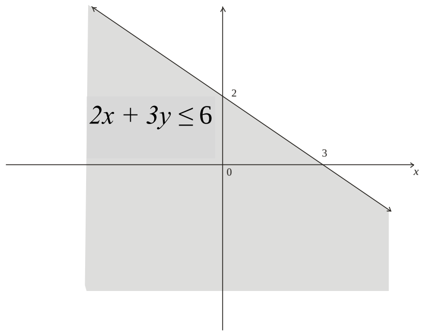
7. $2x + 3y \leq 6, \quad x + 4y \leq 4, \quad x \geq 0, \quad y \geq 0.$

Answers

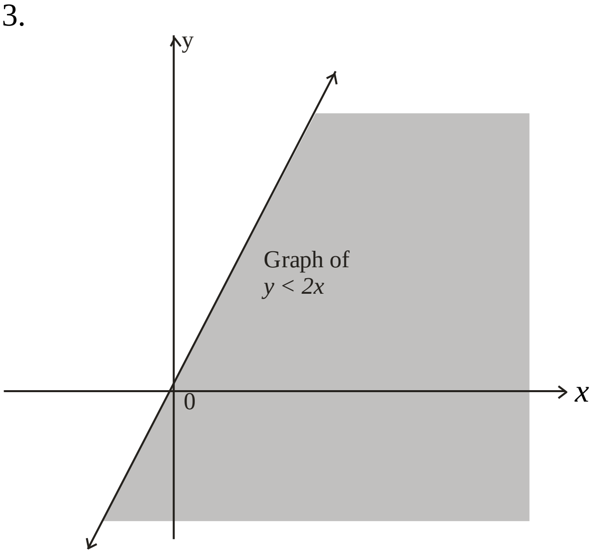
1.



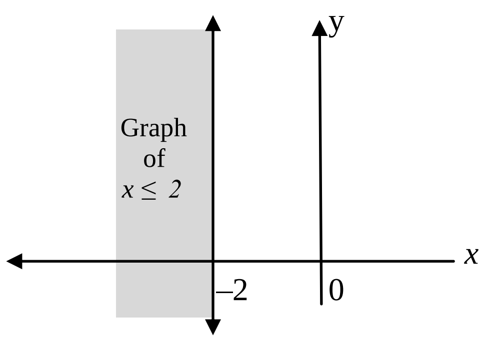
2.



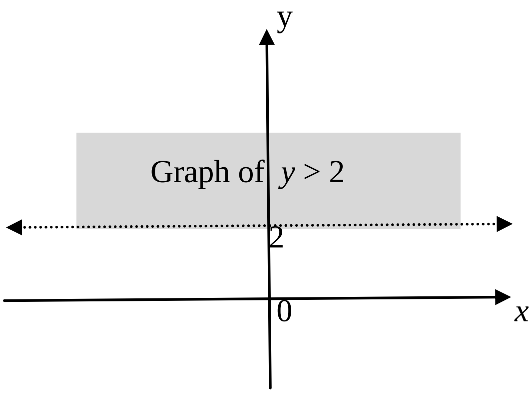
3.

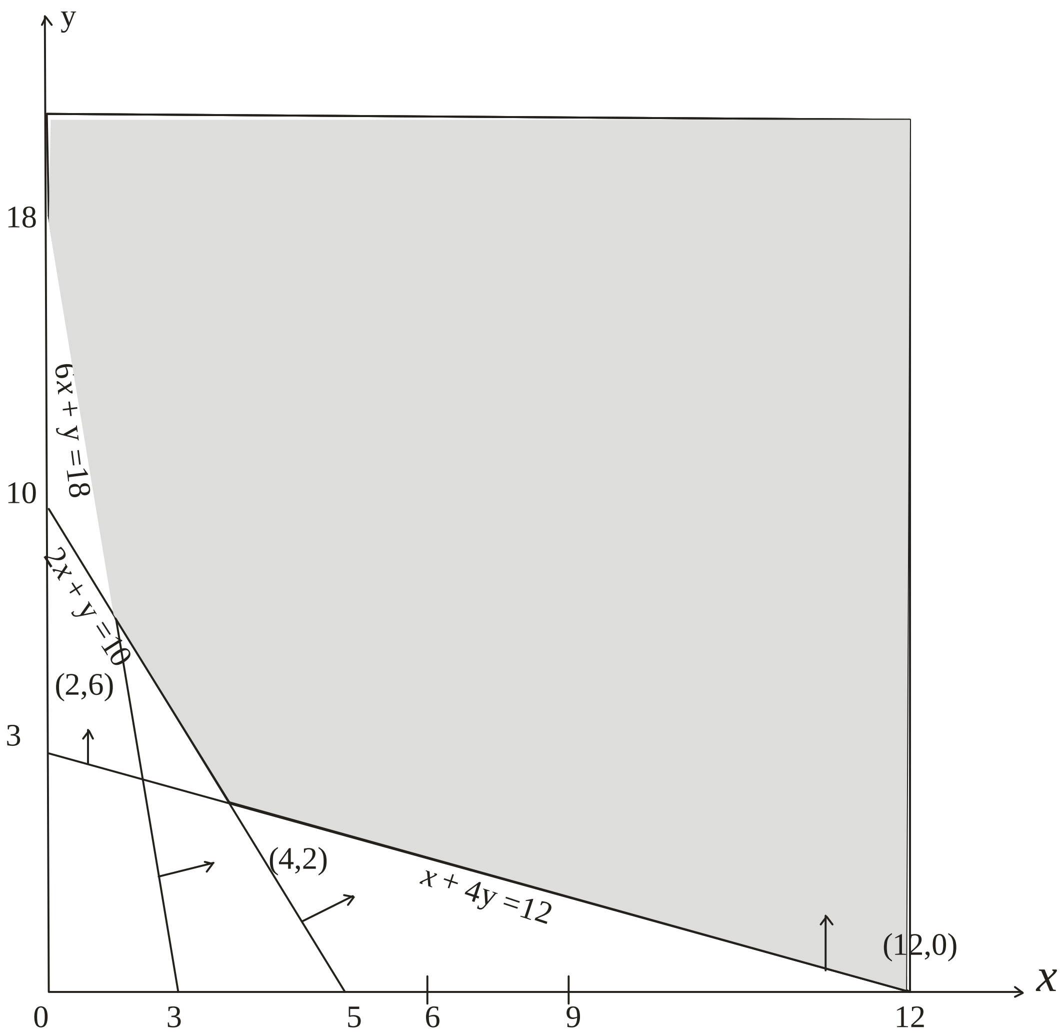


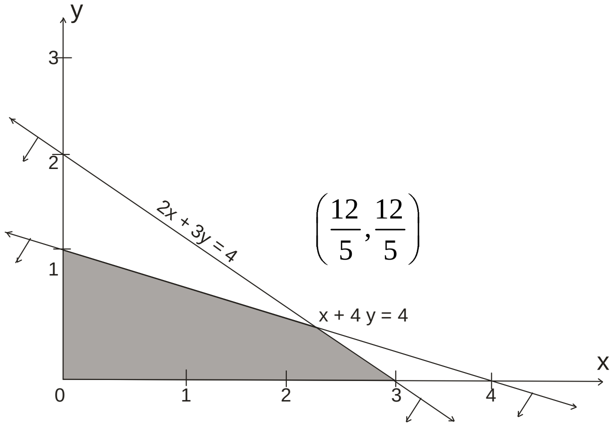
4.



5.


y
Graph of $y > 2$
2
0
x


y
18
10
3
$6x + y = 18$
$2x + y = 10$
(2,6)
(4,2)
$x + 4y = 12$
(12,0)
x
0
3
5
6
9
12


y
3
2
1
0
1
2
3
4
x
2x + 3y = 4
x + 4 y = 4
$\left(\frac{12}{5}, \frac{12}{5}\right)$

4.7 QUADRATIC AND OTHER NON–LINEAR INEQUALITIES

In this section, we examine quadratic inequalities in one variable. Consider the inequality $x^2 - 5x + 6 \leq 0$.

The solution set of this inequality is the set of all real x which make the inequality a true statement.

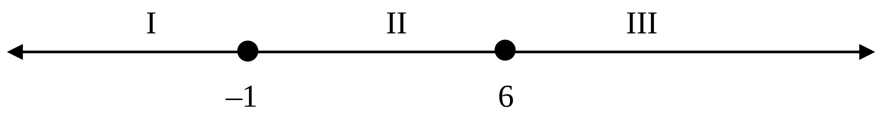
In general, a quadratic inequality is an inequality of the form $ax^2 + bx + c > 0$ where $a \neq 0$. The sign $>$ can be replaced by $\geq$, $<$ or $\leq$.

We illustrate the method of solution of a quadratic inequality by an illustration.

Illustration Solve $x^2 - 5x - 6 < 0$.

First factorise the left–hand side : $(x + 1)(x - 6) < 0$

Next show on a number line the points for which either factor is zero. These points divide into the line into three regions as follows :



Now test each region to see if the inequality is satisfied there. To do this it is sufficient to substitute one value from the each region.

Region I Using $x = -2$ as a test value we see that both $(x + 1)$ and $(x - 6)$ are negative, so that their product is positive. Thus, the inequality $x^2 - 5x - 6 < 0$ is not satisfied in this region. As such we reject region I.

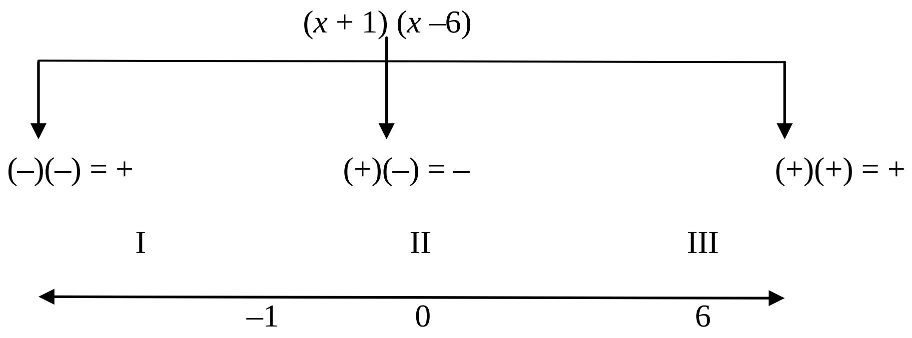
Region II Using $x = 0$ as a test value we see that $(x + 1)$ is positive and $(x - 6)$ is negative. Thus, the inequality $x^2 - 5x - 6 < 0$ is satisfied in this region. That is, this region is a part of the solution.

Region III Using $x = 7$ as a test value, we see that both $(x + 1)$ and $(x - 6)$ are positive, so that their product is positive. Thus, the inequality $x^2 - 5x - 6 < 0$ is not satisfied in the region. As such we reject the region III.

Finally, we check the critical numbers themselves (critical numbers are the number, for which the expression on the left–hand side becomes zero). As $x = -1$ and $x = 6$ make $x^2 - 5x - 6$ equal to zero, these are not parts of the solution. Hence, the solution set is region II :

$$\{x \mid -1 < x < 6\}$$

In practice, we do much of the work mentally and only need to show a sketch as follows :



(It is a straight line)
This shows that the solution set is

$$\{x \mid -1 < x < 6\}$$

CAUTION : Sometimes students attempt to solve a quadratic inequality as if it were a quadratic equation and write

$$(x + 1)(x - 6) < 0$$
$$\Leftrightarrow\ x + 1 < 0 \text{ and } x - 6 < 0$$
$$\Leftrightarrow\ x < -1 \text{ and } x < 6$$

The method is not correct and should never be used.

The above procedure may be used to solve any quadratic inequality and also any other inequality in which one side is zero and other can be written as product and /or quotient of linear factors.

Procedure to Solve a Quadratic Inequality

Step 1 Write the inequality in such a way that one side of the inequality is zero.

Step 2 Write the quadratic expression as product of linear factors.

Step 3 Replace the inequality sing by the equality sign and solve the resulting equation to obtain critical numbers.

Step 4 Plot the critical numbers on the number line and name the different region as I, II, III and so, on.

Step 5 Take a value from each region and check if the inequality is satisfied for that value. If it does include the region into the solution set otherwise reject the region.

Step 6 Check each of the critical to see if it is a part of the solution.

Step 7 Write the solution set.

Example 25 : Solve the following inequalities.

(i) $15\,x^2 + 4x - 4 \geq 0$ (ii) $x^2 + 9x \leq -18$ (iii) $x\,(x+2) < -1$
(iv) $20x - 25 - 4\,x^2 \leq 0$ (v) $x^2 + x + 1 > 0$ (vi) $3x^2 - 8x < 3$

Solution :

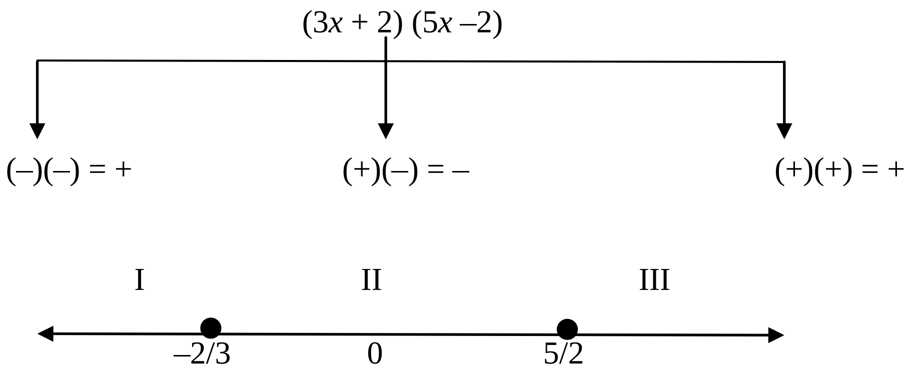
As one side of the inequality is already zero, we go to the second step and resolve the LHS into quadratic factors :

$$15x^2 + 4\,x - 4 = 15\,x^2 + 10\,x - 6x - 4 = 5x\,(3x+2) - 2\,(3x+2)$$

$$= (3x+2)\,(5x-2)$$

In this case, the critical numbers can be obtained by solving $3x + 2 = 0$ and $5x - 2 = 0 \Rightarrow x = -2/3,\ 2/5$.

We now work as explained in the illustration.



Thus, region I and region III are included in the solution set.

For $x = -2/3,\ 3x + 2 = 0$

$\Rightarrow (3x+2)\,(5x-2) = 0 \leq 0$

$\therefore$ –2/3 is included in the solution set.

Similarly, $x = 5/2$ is included in the solution set.

Hence, the solution set of the inequality is

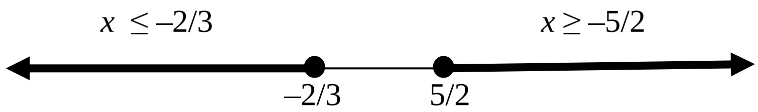
$\{x|x \leq -2/3\} \cup \{x|x \geq 5/2\} \cup \{-2/3,\ 5/2\}$

If we combine $x = -2/3$ with $x < -2/3$, we obtain $x \leq -2/3$ and combine $x = 5/2$ with $x > 5/2$ to obtain $x \geq 5/2$.

Thus, the required solution set is

$\{\,x|x \leq -2/3\,\} \cup \{x|x \geq 5/2\}$

We may represent it on the number line as follows :

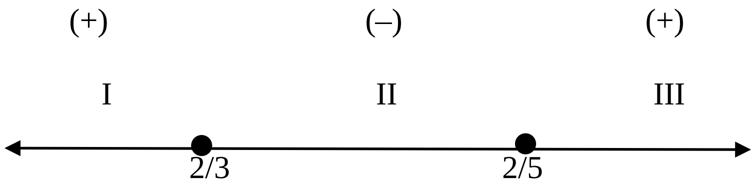


Simplification (In case of Distinct Linear Factors): If one side of the inequality is product of distinct linear factors and the other side is zero, then it is sufficient to know the sign the of the inequality in just one region. The signs in the remaining regions can be obtained by the following rule :

"Two adjacent regions must have opposite signs."

For instance, if we put $x = 0$ in $(3x+2)\ (5x-2)$, we obtain –4 which is negative, Therefore, sign of $(3x+2)\ (5x-2)$ is negative in region II.

By the above rule, the signs in region I and region III must be positive.

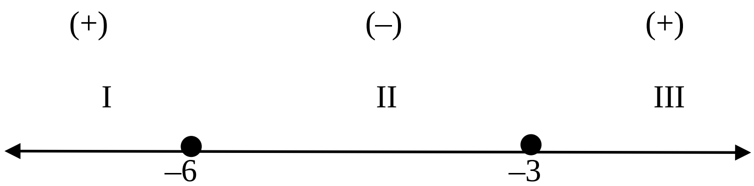


This is the same thing as we obtained earlier.

(ii) We rewrite the inequality as
$x^2 + 9x + 18 \le 0$ or $(x + 6)\ (x + 3) \le 0$

The critical numbers are –6 and –3. As distinct linear factors are involved we use the above simplified procedure.

By taking $x = 0$, we find $(x + 6)(x + 3) = 18 > 0$ and hence has a positive sign in region III. Thus, sign in region II must be negative and hence, in region I it must be positive.



Thus region II is a part of the solution.

Also, $x = -6$ and $x = -3$ satisfy the given inequality.

Therefore, solution of the inequality is

$\{x| -6 < x < -3\} \cup \{-6,-3\} = \{x| -6 \le x < -3\}$

(iii) We rewrite the inequality as $x^2 + 2x + 1$, 0 or $(x + 1)^2 < 0$.

But there is no real x for which $(x + 1)^2 < 0$

Hence, the given inequality has no solution.

(iv) We multiply the inequality by a negative sign to obtain the inequality

$4x^2 - 20x + 25 \geq 0$ or $(2x-5)^2 \geq 0$.

This inequality is satisfied for all real x. Therefore, the solution set of the inequality is **R** is the set of real numbers.

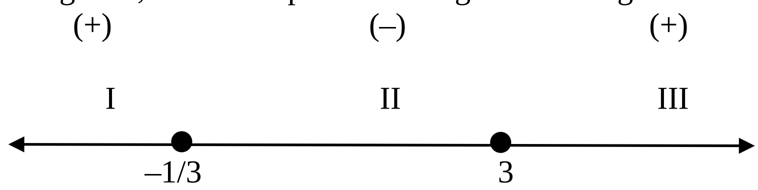
(v) We complete the square using the quadratic and linear factors :

$$x^2 + x + 1 = x^2 + 2\left(\frac{1}{2}\right)x + \frac{1}{4} - \frac{1}{4} + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

As $(x + ½)^2 > 0$ for each real x, we get $x^2 + x + 1 > 0$ for each real x. Hence, solution set of the inequality is **R**

(vi) We rewrite the inequality as $3x^2 - 8x - 3 < 0$ or $(3x + 1)(x-3) < 0$.
The critical numbers are $x = -1/3$ and $x = 3$.

As distinct linear factors are involved, we require the sign of inequality in just one region. We can easily obtain this sign in region II by taking $x = 0$. As the sign in region II is negative, it must be positive in region I and region III.



Also $x = -1/3$ and $x = 3$ do not satisfy the given inequality. Hence, the solution set of this inequality is

$$\{x \mid -1/3 < x < 3\}.$$

Using Absolute Value to Solve Quadratic Inequalities

Sometimes, it is useful to convert a quadratic inequality into an inequality involving absolute value. The following are the two rules :

1. For $a > 0$

$$x^2 > a^2 \Leftrightarrow |x| > a \Leftrightarrow (x < -a \text{ or } x > a)$$

2. For $a > 0$,
$x^2 > a^2 \Leftrightarrow |x| < a \Leftrightarrow -a < x < a$
The above rules hold if we replace '>' by '≥' and '<' by '≤'

CAUTION $x^2 > a^2$ *does not imply* $x > \pm a$

and $x^2 < a^2$ does not imply $x < \pm a$.

Example 26 : Solve the following inequalities.

(i) $9 - x^2 \geq 0$ (ii) $4 - x^2 < 0$

(iii) $x^2 - 4x - 21 \leq 0$ (iv) $(x + 6)^2 > 49$

Solution

(i) $9 - x^2 \geq 0 \Leftrightarrow 9 \geq x^2 \Leftrightarrow x^2 \leq 9 \Leftrightarrow |x| \leq 3 \Leftrightarrow -3 \leq x \leq 3.$

(ii) $4 - x^2 < 0 \Leftrightarrow x^2 > 4 \Leftrightarrow |x| > 2 \Leftrightarrow x < -2$ *or* $x > 2$

(iii)We first complete the squares using first two terms :

$x^2 - 4x - 21 = x^2 - 2(2)x + 2^2 - 21 - 2^2 = (x-2)^2 - 25.$

Now, $x^2 - 4x - 21 \leq 0 \Leftrightarrow |x-2| \leq 5 \Leftrightarrow -5 \leq x - 2 \leq 5 \Leftrightarrow -3 \leq x \leq 7.$

(iv) $(x+6)^2 > 49 \Leftrightarrow (x+6)^2 > 7^2$

$\Leftrightarrow |x+6| > 7 \Leftrightarrow x + 6 < -7$ *or* $x + 6 > 7$

$\Leftrightarrow x < -13$ or $x > 1$

Algebraic Method to Solve Quadratic Inequality

We explain the technique in the following example :

Example 27 : Solve the following quadratic inequalities :

(i) $x^2 + x - 12 \leq 0$ (ii) $6x^2 + 7x - 3 > 0$

Solution

(i) We first factorise the LHS

$x^2 + x - 12 = x^2 + 4x - 3x - 12 = x(x+4) - 3(x+4)$

$= (x+4)(x-3)$

Thus, the give inequality becomes $(x + 4)(x - 3) \leq 0$.

But $ab \leq 0 \Leftrightarrow (a \leq 0, b \geq 0)$ or $(a \geq 0, b \leq 0)$

Therefore, $(x + 4)(x - 3) \leq 0$

$\Leftrightarrow$ $x + 4 \leq 0$ and $x - 3 \geq 0$ *or* $x + 4 \geq 0$ *and* $x - 3 \leq 0$

$\Leftrightarrow$ $x \leq -4$ and $x \geq 3$ *or* $x \geq -4$ and $x \leq 3$

But $x \leq -4$ and $x \geq 3$ is not possible.

Also, $x \leq -4$ and $x \leq 3 \Leftrightarrow -4 \leq x \leq 3$.

Hence, solution set of the given inequality is $\{x \mid -4 \leq x \leq 3\}$.

We first factorise the LHS

$6x^2 + 7x - 3 = 6x^2 + 9x - 2x - 3 = 3x(2x-3) - (2x+3)$

$= (2x+3)(3x-1)$

Thus, the given inequality becomes $(2x + 3)(3x - 1) > 0$

But $ab > 0 \Leftrightarrow (a > 0, b > 0)$ or $(a < 0, b < 0)$

Therefore, $(2x + 3)(3x - 1) > 0$

$\Leftrightarrow$ $2x + 3 < 0$ and $3x - 1 < 0$
or $2x + 3 > 0$ and $3x - 1 > 0$

$\Leftrightarrow$ $x < -3/2$ and $x < 1/3$

or $x > -3/2$ and $x > 1/3$

Now, $x < -3/2$ and $x < 1/3$ can hold simultaneously if $x < -3/2$ whereas $x > -3/2$ and $x > 1/3$ can hold simultaneously if $x > 1/3$.

Thus, the solution set of the inequality is $\{x | x < -3/2 \text{ or } > 1/3\}$.

Check Your Progress 4

Solve the following inequalities :

1. $x^2 + 5x \le -6$
2. $6x - x^2 - 8 \ge 0$
3. $x(x + 4) < -3$
4. $x(x + 2) \ge 0$
5. $x^2 - x + 1 > 0$
6. $2x^2 - 5x + 2 > 0$

Solve the following inequalities by algebraic method.

7 $x^2 - 5x + 6 \ge 0$

8 $3x^2 - 10x + 3 < 0$

4.8 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

1. $2x + 1 > -3 \quad \Leftrightarrow 2x > -1 - 3 \Leftrightarrow 2x > -4 \quad \Leftrightarrow x > -2$

$\therefore$ Solution set of the inequality is $\{x |\, x > -2\} = (-2, \infty)$

Graph of the solution set is

2. $3x - 2 \ge 4 \Leftrightarrow 3x \ge 2 + 4 = 6 \Leftrightarrow 3x \ge 6 \quad \Leftrightarrow x \ge 2$

$\therefore$ Solution set of the inequality is

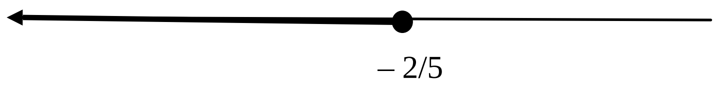
$\{x \,|\, x \ge 2\} = [2, \infty)$

Graph of the solution set is

3. $2 - 5x \le 4 \iff -5x \ge 4 - 2 \iff -5x \ge 2 \iff x \le -2/5\}$

$\therefore$ Solution set is $\{ x \mid x \le -2/5\} = (-\infty, \mathbf{-2/5}]$

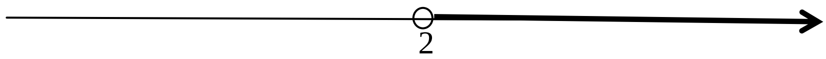
Graph of the solution set is



4. $3 + 2x < 7 \iff 2x < 7 - 3 \iff 2x < 4 \iff x < 2$

$\therefore$ Solution set is $\{ x \mid x < 2\} = (2, -\infty)$

Graph of the solution set is



5. $3x + 2 \le 2x + 5 \iff 3x - 2x \le 5 - 2 \iff x \le 3$

Thus, the solution set is $\{ x \mid x \le 3 \} = [3, \infty)$

6. $3(x - 2) \ge 5 - 2x \iff 3x - 6 \ge 5 - 2x$

$\iff 5x \ge 11 \iff x \ge 11/6$

Thus, the solution set is $\{ x \mid x \ge 11/6 \} = [11/6, \infty)$

7. $-(x - 4) < 2x + 4 \iff -x + 4 < 2x + 4 \iff -3x < 0$

$\iff \boldsymbol{x > 0}$

Thus, the solution set is $\{ x \mid x > 0) = (0, \infty)$

8. Multiply both the side by 15 to obtain

$3(x - 3) \ge 35(3 - x) \iff 3x - 9 \ge 105 - 35x$

$\iff 38x \ge 114 \iff x \ge 114/38 = 3$

Thus, the solution set is $\{ x \mid x \ge 3 \} = [3, \infty)$

9. $-2 \le 5 - 4x \le 7 \iff -2 - 5 \le -4x \le -5 + 7$

$\iff -7 \le -4x \le 2 \iff -2 \le 4x \le 7$

$\iff -2/4 \le x \le 7/4 \iff -1/2 \le x \le 7/4$

Thus, the solution set is $\{x \mid -1/2 \le x \le 7/4\} = [-½, 7/4]$

10. Multiply both the sides of the inequality by 4 to obtain

$-12 \leq x - 2 \leq 16 \Leftrightarrow -12 + 2 \leq x \leq 16 + 2$

$\Leftrightarrow -10 \leq x \leq 18$

Thus, the solution set is $\{ x \mid -10 \leq x \leq 18 = [-10, 18]$

11. $\dfrac{x+3}{x-1} \leq 0 \Leftrightarrow x + 3$ and $x - 1$ must be of opposite signs.

Case 1 $x + 3 < 0$ and $x - 1 > 0$

$\Leftrightarrow x < -3$ and $x > 1$

No such x exists.

Case 2 $x + 3 > 0$ and $x - 1 < 0$

$\Leftrightarrow x > -3$ and $x < 1$

Thus, $-3 < x < 1$

$\therefore$ Solution set of the inequality is $(-3, 1)$

12. $\dfrac{x+8}{x+1} > 1 \Leftrightarrow \dfrac{x+8}{x+1} - 1 > 0$

$\Leftrightarrow \dfrac{x+8-x-1}{x+1} > 0 \Leftrightarrow \dfrac{7}{x+1} > 0.$

Now, $\dfrac{7}{x+1} > 0,\quad 7 > 0 \Leftrightarrow x + 1 > 0, \quad \Leftrightarrow x > -1$

Thus, solution set of the inequality is $(-1, \infty)$

Check Your Progress 2

1. $|x-3| \geq 2 \Leftrightarrow x - 3 \leq -2$ or $x - 3 \geq 2$

$\Leftrightarrow x \leq 1 \; or \; x \geq 5$

Thus, Solution set is $(-\infty, 1] \cup [5, \infty)$

2. $\left|\frac{2x-3}{3}\right| \leq \Leftrightarrow -1 \leq \frac{1}{3}(2x-5) \leq 1$

$\Leftrightarrow \quad -3 \leq 2x - 5 \leq 3 \qquad \Leftrightarrow \quad 2 \leq 2x \leq 8$
$\Leftrightarrow 1 \leq x \leq 8/3$

Thus, Solution set is $[1, 8/3]$

3. $\left|\frac{x-5}{3}\right| < 6 \quad \Leftrightarrow \; -6 < \frac{x-5}{3} < 6$

$\Leftrightarrow -18 < x - 5 < 18 \quad \Leftrightarrow \quad -13 < x < 23$

$\therefore$ Solution set is $(-13, 23)$

4. $\frac{5}{|x-3|} < 7 \;\Leftrightarrow\; 5 < 7|x-3|, x \neq 3$

$\Leftrightarrow \; |x-3| > \frac{5}{7}, x \neq 3$

$\Leftrightarrow \; x-3 < -\frac{5}{7}$ or $x-3 > \frac{5}{7}, x \neq 3$

$\Leftrightarrow \; x < \frac{16}{7}$ or $x > 26/7, x \neq 3$

Thus, Solution set is $(-\infty, 16/7) \cup (\frac{26}{7}, \infty) - \{3\}$
$= (-\infty, 16/7) \cup (\frac{26}{7}, \infty)$.

5. We wish to find x so that $200 < T < 300$, that is,

$200 < 30 + 25(x-3) < 300$

$\Leftrightarrow \quad 200 - 30 < 25(x-3) < 300 - 30 \Leftrightarrow \quad 170 < 25(x-3) < 270$
$\Leftrightarrow \quad 170/25 < x - 3 < 270/50 \Leftrightarrow \; 6.8 + 3 < x < 10.8 + 3$

$\Leftrightarrow \quad 9.8 < x < 13.8$

Thus, the required depth lies between 9.8 and 13.8 km.

6. We are given CA = 12 and $80 \leq IQ \leq 140$. Thus,

$80 \leq \frac{MA}{12}(100) \leq 140 \;\Leftrightarrow\; \frac{80 \times 12}{100} \leq MA \leq \frac{140 \times 12}{100}$

$\Leftrightarrow \; 9.6 \leq MA \leq 16.8$

7. Profit = Revenue – Cost
= $300x - (3000 + 200x)$
= $100x - 3000$

In order to have a profit, we must have

Profit > 0

$\Leftrightarrow$ $100\,x - 3000 > 0$

$\Leftrightarrow$ $100\,x > 3000$

$\Leftrightarrow$ $x > 30$

As x is a natural number, $x \geq 31$

Thus, Josh mobiles must manufacture at least 31 mobile sets.

8. Let m = amount of medication (in CC) given to the patient,

   We must have

   $| m - 3 | < 0.005$

   $\Leftrightarrow - 0.005 < m - 3 < 0.005$

   $\Leftrightarrow$ $2.995 < m < 3.005$

Check Your Progress 3

1. The double – intercept form of the boundary is $\frac{x}{-3} + \frac{y}{3} = 1$. Test point (0,0) satisfies the inequality. The graph is given in the answers.

2. The double – intercept form of the boundary is $\frac{x}{3} + \frac{y}{2} = 1$. Test point (0,0) satisfies the inequality. The graph is given in the answers.

3. The boundary $y = 2x$ represents the line through the origin and (1,2). Note that (0,2) does not satisfy the inequality $y < 2x$. So, shade the half plane not containing (0,2).

4. $x = -2$ is a vertical line through $(-2, 0)$.

5. $y = 2$ is a horizontal line through (0,2).

6. Since $x \geq 0$ and $y \geq 0$ we restrict ourselves to the first quadrant only. We draw the boundaries.

   $2x + y = 10 \left(i.e., \frac{x}{5} + \frac{y}{10} = 1\right)$, $x + 4y = 12 \left(i.e., \frac{x}{12} + \frac{y}{3} = 1\right)$,

   and $6x + y = 18 \left(i.e., \frac{x}{3} + \frac{y}{18} = 1\right)$

   Next, we indicate the solution set of each of the inequalities by drawing the arrows, and then finally shade the common region (i.e., the solution set).

7. We once again restrict ourselves to the first quadrant. We draw the boundaries.

   $2x + 3y = 6 \left(i.e., \frac{x}{3} + \frac{y}{2} = 1\right)$ and $x + 4y = 4 \left(i.e., \frac{x}{4} + \frac{y}{1} = 1\right)$.
   We indicate the solution set of each of the inequalities by arrows and shade the common region, i.e., the solution set.

Check Your Progress 4

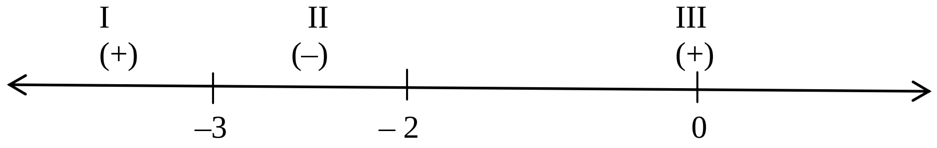
1 $x^2 + 5x \leq -6 \qquad \Leftrightarrow x^2 + 5x + 6 \leq 0$

$\Leftrightarrow x^2 + 2x + 3x + 6 \leq 0 \quad \Leftrightarrow x\ (x+2) + 3\ (x+2) \leq 0$

$\Leftrightarrow (x+3)(x+2) \leq 0$

The critical numbers are –3 and – 2

As distinct linear factors are involved, we required sign of the inequality in one of the three regions. For $x = 0$, $x^2 + 5x + 6 = 6 > 0$. Signs in different regions are shown below.



Thus, solution set is [– 3, –2]

2. $x^2 - 6x + 8 \leq 0$

$\Leftrightarrow x^2 - 6x + 8 \leq 0 \qquad \Leftrightarrow x^2 - 2x - 4x + 8 \leq 0$

$\Leftrightarrow x\,(x-2) - 4\,(x-2) \leq 0 \qquad \Leftrightarrow (x-2)(x-4) \leq 0$

As explained in Question 1, $2 \leq x \leq 4$.

See the figure below.

Thus, solution set is [2,4]

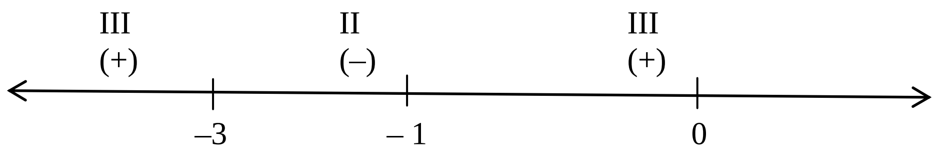
3. $x\,(x+4) < -2 \quad \Leftrightarrow x^2$

$\Leftrightarrow x^2 + 3x + x + 3 < 0 \qquad \Leftrightarrow x(x+3) + (x+3) < 0$

$\Leftrightarrow (x+3)\,(x+1) < 0$

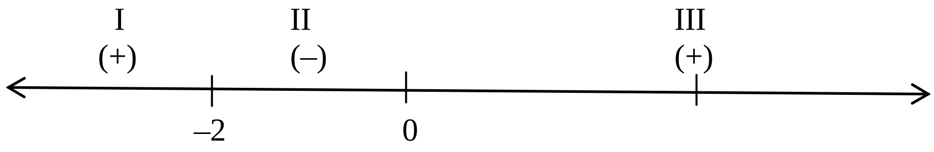
Critical numbers are – 3 and – 1.

Sign of the expression $x^2 - 4x + 3$ in different regions in given below.



Required solution set is (–3, –1)

4. Critical numbers are –2 and 0 sign of $x(x + 2)$ in different regions is shown below.



Required solution set is $(-\infty, -2] \cup [0, \infty)$

5. $x^2 - x + 1 = x^2 - 2\left(\frac{1}{2}\right)x + \frac{1}{4} + 1 - \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} 0$

For each real x.

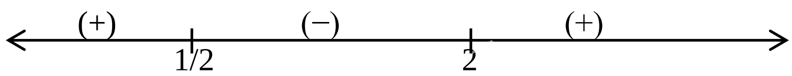
Thus, solution set is $\boldsymbol{R}$

6. $2x^2 - 5x + 2 = 2x^2 - 4x - x + 2$

$= 2x(x - 2) - (x - 2)$

$= (2x - 1)(x - 2)$

Critical numbers are ½ and 2.



$2x^2 - 5x + 2 > 0 \quad \Leftrightarrow \quad x < ½$ or $x > 2$

$\therefore$ Solution set is $\left(-\infty, \frac{1}{2}\right) \cup (2, \infty)$

7. $x^2 - 5x + 6 = x^2 - 2x - 3x + 6 = x(x - 2) - 3(x - 2) = (x - 2)(x - 3)$

Now, $x^2 - 5x + 6 \geq 0$

$\Leftrightarrow (x - 2)(x - 3) \geq 0$

$\Leftrightarrow x - 2 \geq 0, \ x - 3 \geq 0$ or

$x - 2 \leq 0, x - 3 \leq 0$

$\Leftrightarrow (x \geq 2, x \geq 3)$ or$(x \leq 2, x \leq 3)$

$\Leftrightarrow x \geq 3 \ or \ x \leq 2.$

Thus, solution set is $(-\infty, 2) \cup (3, \infty)$

8. $3x^2 - 10x + 3 = 3x^2 - 9x - x + 3$

$= 3x(x - 3) - (x - 3)$

$= (3x - 1)(x - 3)$

Now, $3x^2 - 10x + 3 < 0$

$\Leftrightarrow (3x - 10)(x - 3) < 0$

$\Leftrightarrow (3x - 1 < 0, x - 3 > 0) \ or \ (3x - 1 > 0, x - 3 < 0)$

$\Leftrightarrow$ ($x < 1/3$, $x > 3$) or ($x > 1/3$, $x > 3$)

$$\Leftrightarrow \frac{1}{3} < x < 3$$

Thus, solution set is (1/3, 3).

4.9 SUMMARY

In this unit, to begin with, in **section 4.2**, a number of tools for solving inequalities are given. Then, a number of examples are given to illustrate how to use these tools in solving inequalities in one variable and also how to draw graph of the solution, on the number line. In **section 4.3**, solutions of inequalities involving absolute values are discussed. In **sections 4.4, 4.5 and 4.6**, first the concept of linear inequalities in two variables, is discussed and then method of solving these inequalities graphically, are explained with suitable examples. In **section 4.7**, some methods of solving quadratic and other non-linear inequalities, are first discussed and then illustrated with suitable examples.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 4.8**.